The structure of decomposition of a triconnected graph

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Introduction

The structure of decomposition of a connected graph by its cutpoints (i.e., vertices which deleting makes graph disconnected) is well known [1, 2]. It is convenient to describe this structure with the help of so-called *tree of blocks and cutpoints*. The vertices of this tree are cutpoints and parts of decomposition of the graph by its cutpoints.

In 1966 W. T. Tutte [3] described the structure of relative disposition of 2-vertex cutsets in biconnected graphs and showed, that this structure has much in common with the the structure of cutpoints. In particular, a construction of the tree of blocks for biconnected graph was introduced. Analogous constructions were considered also in the works [4, 5].

Attempts of development of analogous constructions for graphs of greater connectivity were done in the works [6, 7, 8]. But significant difficulties appear in this process. They are concerned with the fact that two k-vertex cutsets of a k-connected graph can be dependent, i.e. after deleting of one of them, vertices of another could appear in different connected components. This leads to non-uniqueness of resulting constructions of a tree of blocks for k-connected graphs. These constructions are essentially dependent on the order in which cutsets were chosen during the process. Moreover, such constructions take account of not all k-vertex cutsets: splitting the graph by one cutset we automatically loose information about all cutsets dependent with the chosen one. In the works [7, 8] these difficulties were partly overcome for graphs, satisfying some additional condition. But in general case the question of how to describe the structure of decomposition of a k-connected graph by its k-vertex cutsets for $k \geq 3$ remained open.

In the work [9] it was developed a new method of studying of the structure of relative disposition of k-vertex cutsets of a k-connected graph — the theorem of decomposition. With the help of this theorem several results for the case of arbitrary k were obtained. As an illustration of work of

new method in the end of the work [9] one can see rather simple and visual description of the structure of 2-vertex cutsets of a biconnected graph. In general, this description is similar to the construction of Tutte [3], but it is a good illustration of efficiency of the new method.

This paper is devoted to studying of the structure of relative disposition of 3-vertex cutsets in a (vertex) triconnected graph. We will use the theorem of decomposition and as a result we obtain a description, similar to analogous structure of a biconnected graph [3, 9].

1 Basic notations

Always in our paper we consider simple undirected finite graphs without loops and multiple edges.

We use the following notations and definitions. For a graph G we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. We denote the *degree* of a vertex x in the graph G by $d_G(x)$.

We call two vertices *connected*, if there is a path between them. By a *connected component* of a graph we always mean its maximal (with respect to inclusion) set of pairwise connected vertices.

A graph G is called *k-connected*, if it contains at least $k+1$ vertices and remains connected after deleting any $k-1$ vertices. In particular, for $k=2$ such a graph is called *biconnected*, and for $k=3$ — *triconnected*.

For any set of edges $E \subset E(G)$ we, as usual, denote by $G-E$ the graph obtained from G after deletion of edges of the set E. For $e \in E(G)$ we set $G-e = G-\{e\}$.

For any set of vertices $V \subset V(G)$ we denote by $G-V$ the graph obtained from G after deletion of vertices of the set V and all edges incident to deleted vertices. For $v \in V(G)$ we set $G-v$=$G-\{v\}$.

For any set $M \subset V(G) \cup E(G)$ we denote by $G-M$ the graph obtained from G after deletion of all vertices and edges of the set M and all edges incident to deleted vertices.

During all our work let G be a triconnected graph with $|V(G)| > 6$.

A set $S \subset V(G)$ is called a *cutset*, if the graph $G-S$ is disconnected. We denote the set of all cutsets of the graph G by $\mathfrak{R}(G)$, and the set of all 3-vertex cutsets of G (we will call them simply 3-*cutsets*) — by $\mathfrak{R}_3(G)$.

We use terminology of the work [9]. We rewrite definitions from [9], that we need, in the form convenient for triconnected graphs.

Definition 1. 1) Let $R, X \subset V(G)$. We say that R *splits* X, if not all vertices of the set $X \setminus R$ are in the same connected component of the graph $G-R$.

2) Let $U, W \subset V(G)$. We say that R *separates* a set U from a set W, if $U \not\subset R$, $W \not\subset R$ and any two vertices $u \in U \setminus R$ and $w \in W \setminus R$ lie in different connected components of the graph $G - R$.

In the case $U = \{u\}$ we say that R separate a vertex u from a set W. If $U = \{u\}$ and $W = \{w\}$, we say that R separates a vertex u from a vertex w.

Definition 2. 1) We call sets $S, T \in \mathfrak{R}_3(G)$ *independent*, if S does not split T and T does not split S. Otherwise, we call these sets *dependent*.

2) We assign to each set $\mathfrak{S} \subset \mathfrak{R}_3(G)$ the *dependence graph* $\mathrm{Dep}(\mathfrak{S})$, which vertices are cutsets of $\mathfrak{S}$, and two vertices are adjacent if and only if correspondent cutsets are dependent.

Thus, any set $\mathfrak{S}$ is divided into *dependence components* — subsets, correspondent to connected components of the graph $\mathrm{Dep}(\mathfrak{S})$.

It is easy to prove, that if T does not split S, then S does not split T, i.e. these cutsets are independent (see [6, 7]).

Definition 3. Let $\mathfrak{S} \subset \mathfrak{R}_3(G)$.

1) A *part* of decomposition of the graph G by the set $\mathfrak{S}$ (or a part of $\mathfrak{S}$-decomposition) is a maximal (with respect to inclusion) set $A \subset V(G)$ such, that no cutset $S \in \mathfrak{S}$ splits A. We denote by $\mathrm{Part}(\mathfrak{S})$ the set of all such parts. If $\mathfrak{S}$ consists of one cutset S, then we denote the set of all parts of $\{S\}$-decomposition by $\mathrm{Part}(S)$.

2) Let $A \in \mathrm{Part}(\mathfrak{S})$. We call *inner vertices* all vertices of A which do not belong to any cutset of $\mathfrak{S}$. The set of all inner vertices of the part A we call *interior* of the part A and denote by $\mathrm{Int}(A)$.

We call *boundary vertices* all vertices of the part A belonging to any cutsets from $\mathfrak{S}$. The set of all such vertices we call *boundary* of the part A and denote by $\mathrm{Bound}(A)$.

3) We call a part A *empty*, if $\mathrm{Int}(A) = \varnothing$ and *nonempty* otherwise. We call a part A *small*, if $|A| < 3$ and *normal*, if $|A| \geq 3$.

It is easy to see, that two different parts $A_1, A_2 \in \mathrm{Part}(\mathfrak{S})$ either have no common vertices, or $A_1 \cap A_2$ is a subset of one of cutsets of $\mathfrak{S}$. It is proved in [9, theorem 2], that $\mathrm{Bound}(A)$ consists of all vertices of a part A, which are adjacent to vertices outside A and $\mathrm{Bound}(A)$ separates $\mathrm{Int}(A)$ from $V(G) \setminus A$.

An important particular case of decomposition of triconnected graph by a set of 3-cutsets is a decomposition by one 3-cutset S. It is clear, that for any part $F \in \mathrm{Part}(S)$ its interior $\mathrm{Int}(F)$ is a connected component of the graph $G - S$.

Since no subset of the cutset S is a cutset of the graph G, then any vertex of S is adjacent to at least one vertex of $\mathrm{Int}(F)$, hence, the induced subgraph of the graph G on the vertex set F is connected.

Note, that any vertex x of the graph G is adjacent to at least one other vertex y. Obviously, no cutset can separate x from y, thus, for any set $\mathfrak{S} \subset \mathfrak{R}_3(G)$ any part $A \in \mathrm{Part}(\mathfrak{S})$ contains at least two vertices. Hence, any small part contains exactly two vertices.

1.1 Dependent and independent cutsets

Let $S, T \in \mathfrak{R}_3(G)$. Clearly these cutsets are independent if and only if there exists a part $F \in \mathrm{Part}(S)$ which contains T. It was proved in [9, lemma 1] that if there exists a part $A \in \mathrm{Part}(S)$ such that $\mathrm{Int}(A) \cap T = \varnothing$, then the cutsets S and T are independent. Decomposition of the graph by a pair of dependent cutsets is described in the following lemma.

Lemma 1 ([9, lemma 7]). *Let G be a k-connected graph and cutsets $S, T \in \mathfrak{R}_k(G)$ be dependent. Let $\mathrm{Part}(S) = \{F_1, \ldots, F_n\}$ and $\mathrm{Part}(T) = \{H_1, \ldots, H_m\}$. For all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ we set*

$$P = S \cap T, \quad S_j = S \cap \mathrm{Int}(H_j), \quad T_i = T \cap \mathrm{Int}(F_i), \quad G_{i,j} = F_i \cap H_j.$$

Then

$$\mathrm{Part}(\{S, T\}) = \{G_{i,j}\}_{i \in \{1,\ldots,n\},\, j \in \{1,\ldots,m\}}, \qquad \mathrm{Bound}(G_{i,j}) = P \cup T_i \cup S_j,$$

moreover, $T_i \neq \varnothing$ for all $i \in \{1, \ldots, n\}$ and $S_j \neq \varnothing$ for all $j \in \{1, \ldots, m\}$.

The statement of lemma 1 is correct for k-cutsets of a k-connected graph for all k. In the case $k = 3$, which is interesting to us, it is easy to derive the following statements from this lemma. (Notations are the same as in the lemma).

Corollary 1. *Let cutsets $S, T \in \mathfrak{R}_3(G)$ be dependent. Then $|S \cap T| \leq 1$, and each of these cutsets splits G into not more than 3 parts.*

Proof. It is easy to see, that $m, n \leq 3$, since all the sets T_i and S_j are nonempty. Obviously, $m, n \geq 2$, hence, $|P| \leq 1$. $\square$

Corollary 2. *If $S \cap T = \varnothing$, then $\mathrm{Part}(\{S, T\})$ contains at least one small part. $|G_{i,j}| = 2$ if and only if $|T_i| = |S_j| = 1$. Any small part $G_{i,j} \in \mathrm{Part}(\{S, T\})$ consists of two vertices u and v, where $u \in T$ and $v \in S$. Moreover, v is the only vertex of the part H_j adjacent to u, and u is the only vertex of the part F_i adjacent to v.*

Proof. Since $|T_1| + |T_2| \leq 3$, at least one of the sets T_1 or T_2 consists of one vertex. Without loss of generality we may assume, that it is T_1. Analogously, we may assume, that $|S_1| = 1$. Then $|\mathrm{Bound}(G_{1,1})| = |T_1 \cup S_1| = 2$. Hence, $\mathrm{Int}(G_{1,1}) = \varnothing$ and $|G_{1,1}| = 2$. Similarly, each part $G_{i,j}$ for which $|T_i| = |S_j| = 1$ is small.

Let $|G_{i,j}| = 2$. Obviously, then $|T_i| = |S_j| = 1$ and $G_{i,j} = \{u, v\}$, where $T_i = \{u\}$, $S_j = \{v\}$. Since $u \in T$, then the vertex u must be adjacent to at least one inner vertex of the part H_j. On the other side, since $u \in \mathrm{Int}(F_i)$, then the vertex u can be adjacent only to vertices of the part F_i. As well, $F_i \cap H_j = G_{i,j} = \{u, v\}$, hence, v is the only vertex of the part H_j which can be adjacent to u. Thus, the vertices u and v are adjacent and v is the only vertex of the part H_j adjacent to u. Analogously, u is the only vertex of the part F_i adjacent to v. $\square$

Corollary 3. *If $|S \cap T| = 1$, then $m = n = 2$ and $\mathrm{Part}(\{S, T\})$ contains no small parts. Any empty part of $\mathrm{Part}(\{S, T\})$ consists of exactly three vertices u, v and p, where $u \in S \setminus T$, $v \in T \setminus S$ and $P = \{p\}$. Moreover, the vertices u and v are adjacent.*

Proof. Since all sets T_1, T_2, S_1, S_2, P are nonempty, we obtain $m = n = 2$ and $|T_1| = |T_2| = |S_1| = |S_2| = |P| = 1$. Any part $G_{i,j}$ must contain at least one vertex from the sets T_i, S_j and P, i.e., at least 3 vertices. Thus, there are no small parts.

If $\mathrm{Int}(G_{i,j}) = \varnothing$, then $G_{i,j} = \mathrm{Bound}(G_{i,j}) = T_i \cup S_j \cup P$. Hence, $|G_{i,j}| = 3$. Let $T_i = \{u\}$, $S_j = \{v\}$, $P = \{p\}$. Then $G_{i,j} = \{u, v, p\}$. We can prove, that the vertices u and v are adjacent, as well as in corollary 2. $\square$

Remark 1. We can exclude the case $m = n = 3$, because in this case $P = \varnothing$ and $|T_1| = |T_2| = |T_3| = |S_1| = |S_2| = |S_3| = 1$. Obviously, then all parts of $\mathrm{Part}(\{S, T\})$ are small. Thus, $V(G)$ consists of vertices of the sets S and T, i.e. $|V(G)| = 6$. As it was written in the beginning of our paper, we do not consider such graphs. (It is easy to see, that in this case the graph G is isomorphic to $K_{3,3}$.)

Lemma 2. *Let sets $S, T \in \mathfrak{R}_3(G)$ and parts $A_1, \ldots, A_k \in \mathrm{Part}(T)$ be such that $S \cap \mathrm{Int}(A_i) = \varnothing$ for $i \in \{1, \ldots, k\}$. Then the set S does not split $A = \cup_{i=1}^{k} A_i$.*

Proof. It is easy to see, that vertices of each of sets $\mathrm{Int}(A_1), \ldots, \mathrm{Int}(A_k)$ are connected in $G - S$. Every vertex of a nonempty set $T \setminus S$ is adjacent to a vertex of each set $\mathrm{Int}(A_1), \ldots, \mathrm{Int}(A_k)$. Hence, S does not split A. $\square$

2 Basic structures

In this section we describe basic structures, which dependent cutsets can form, and investigate basic properties of these structures.

2.1 Flowers in triconnected graphs

Consider a tuple $F = (p; q_1, \dots, q_m)$ of vertices of our graph G (here $m \geq 4$), in which the vertices $q_1, \dots, q_m$ are *cyclic ordered*. We will set, that a cyclic permutation of the set $q_1, \dots, q_m$ does not change the tuple F. Let us introduce the notation $Q_{i,j} = \{q_i, q_j, p\}$. Let $\mathfrak{R}(F)$ consist of sets $Q_{i,j}$ for all pairs of different and non-neighboring in the cyclic order indexes i and j.

Definition 4. We say, that a tuple $F = (p; q_1, \dots, q_m)$ is a *flower*, if there exists such a set $\mathfrak{S} \subset \mathfrak{R}(F)$ that the decomposition $\mathrm{Part}(\mathfrak{S})$ consists of m parts $G_{1,2}$, $G_{2,3}$, $\dots$, $G_{m,1}$ and $\mathrm{Bound}(G_{i,i+1}) = Q_{i,i+1}$.

We call the vertex p the *center*, and the vertices $q_1, \dots, q_m$ — *petals* of this flower. The set $V(F) = \{p, q_1, \dots, q_m\}$ is called the *vertex set* of F. All sets $Q_{i,j} = \{q_i, q_j, p\}$ are called *sets of a flower F*.

We say, that the set $\mathfrak{S}$ *generates* the flower F.

The notations introduced above will be standard for a flower. We will always write petals of a flower in cyclic order and consider their indexes as residues modulo number of petals. Set the notation $G_{i,j} = \cup_{x=i}^{j-1} G_{x,x+1}$ (index x run over all residues from i to $j-1$ in the cyclic order). We suppose, that $G_{x,x} = \varnothing$.

It is proved in [9, lemma 9], that the dependence graph of any set of k-cutsets which generates a flower is connected. It is also proved in [9] (see theorem 6 and corollaries of it), that any set $Q_{i,j}$ separates $G_{i,j}$ from $G_{j,i}$, and, moreover, $\mathrm{Part}(Q_{i,j}) = \{G_{i,j}, G_{j,i}\}$ if $j \notin \{i, i+1, i-1\}$.

We call the sets $Q_{1,2}, Q_{2,3}, \dots, Q_{m,1}$ *boundaries*, and other sets $Q_{i,j}$ — *inner sets* of the flower F.

Let $\mathrm{Part}(F) = \{G_{1,2}, G_{2,3}, \dots, G_{m,1}\}$ be the *decomposition* of the graph G by the flower F. Obviously, no one of these parts is small.

If a part $G_{i,i+1}$ is empty, then $Q_{i,i+1}$ is not a cutset. If $G_{i,i+1}$ is nonempty, then $Q_{i,i+1}$ is a cutset, $G_{i+1,i} \in \mathrm{Part}(Q_{i,i+1})$ and $G_{i,i+1}$ is the union of all different from $G_{i+1,i}$ parts of $\mathrm{Part}(Q_{i,i+1})$.

Lemma 3. *Let a set $\mathfrak{S}$ generate a flower F. Then the intersection of all cutsets of $\mathfrak{S}$ consists of one vertex — the center of the flower F.*

Proof. Let $F = \{p; q_1, \dots, q_m\}$, $\mathfrak{S} = \{S_1, \dots, S_n\}$, $P = \cap_{i=1}^{n} S_i$. It is obvious from $\mathfrak{S} \subset \mathfrak{R}(F)$, that $P \ni p$. If $P \neq \{p\}$, then P contains a petal of

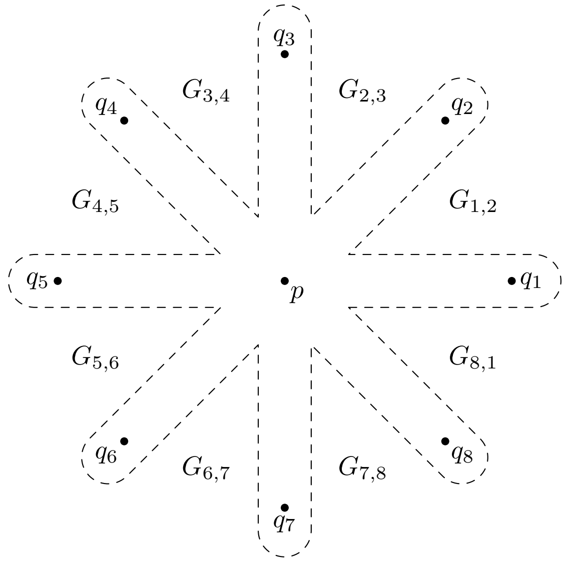

Figure 1: Decomposition of a graph by a flower with eight petals

F, let $P \ni q_i$. Then all parts of $\mathrm{Part}(\mathfrak{S}) = \mathrm{Part}(F)$ contain q_i. However, the set $G_{i+1,i-1} \not\ni q_i$ is a union of several parts of $\mathrm{Part}(F)$. This contradiction finishes the proof. $\square$

Remark 2. 1) If a part $G_{i,i+1}$ is empty, then according to the structure of a flower, described above, by corollary 3 the vertices q_i and q_{i+1} are adjacent.

2) If both parts $G_{i-1,i}$ and $G_{i,i+1}$ are empty, then the vertex q_i is adjacent in the graph G to the vertices p, q_{i-1}, q_{i+1} and only them.

3) If sets $S = \{a, u, v\}$ and $T = \{a, x, y\} \in \mathfrak{R}_3(G)$ are dependent, then with the help of lemma 1 and corollary 3 it is easy to see, that these two sets generate a flower with the center a and cyclic ordered petals u, x, v, y.

A flower F can be generated by different sets of 3-cutsets, however, it is proved in [9, theorem 7], that all such sets split the graph identically — into the parts of $\mathrm{Part}(F)$. Moreover, it is proved in the mentioned theorem, that if sets $\mathfrak{S}$, $\mathfrak{T} \in \mathfrak{R}_k(G)$ generate flowers with the same center and the same set of petals, then these two flowers coincide (i. e. the cyclic orders of petals in these flowers coincide).

On the figure 1 decomposition of a graph by a flower with eight petals it is shown.

In this sense a flower is alike a wheel, from which, according to the work [10], were “originated” all triconnected graphs.

We need the following theorem, also proved in the work [9].

Theorem 1 ([9, theorem 8]). *For any set $\mathfrak{S} \subset \mathfrak{R}_3(G)$ two following statements are equivalent.*

1° *Every part of* $\mathrm{Part}(\mathfrak{S})$ *contains at least three vertices.*

2° *Every dependence component of the set $\mathfrak{S}$ either consists of one cutset, or generates a flower.*

Corollary 4. *If dependence graph of a set $\{S_1, S_2, \dots, S_n\} \subset \mathfrak{R}_3(G)$ is connected and $\cap_{i=1}^n S_i \neq \varnothing$, then this set generates a flower.*

Proof. Obviously, we can enumerate the cutsets of our set such, that for every $\ell \in \{1, \dots, n\}$ the dependence graph of the set $\mathfrak{S}_\ell = \{S_1, S_2, \dots, S_\ell\}$ is connected. Prove by induction on ℓ, that the set $\mathfrak{S}_\ell$ generates a flower. The base for $\ell = 2$ is obvious by remark 2.

Induction step from ℓ to $\ell + 1$. Let the set $\mathfrak{S}_\ell$ generate a flower $F = (p; q_1, \dots, q_m)$. If $S_{\ell+1}$ does not split any part of $\mathrm{Part}(F)$, then $\mathrm{Part}(\mathfrak{S}_{\ell+1}) = \mathrm{Part}(F)$. This decomposition contains no small parts and by theorem 1 the step is proved.

Let $S_{\ell+1}$ split some part of $\mathrm{Part}(F)$. It follows from remark 2, that the cutset $S_{\ell+1}$ cannot split an empty part of $\mathrm{Part}(F)$. Let $S_{\ell+1}$ split a nonempty part $G_{i,i+1}$. Then $S_{\ell+1}$ is dependent with $Q_{i,i+1}$. By lemma 3 we have $\cap_{i=1}^{\ell} S_i = \{p\}$, hence, $S_{\ell+1} \ni p$. Now it follows from dependence of the sets $S_{\ell+1}$ and $Q_{i,i+1}$, that $S_{\ell+1} \cap Q_{i,i+1} = \{p\}$, the intersection $S_{\ell+1} \cap \mathrm{Int}(G_{i,i+1})$ consists of a single vertex x, and vertices q_i and q_{i+1} lie in different parts of $\mathrm{Part}(S_{\ell+1})$. Then by lemma 1 the set $S_{\ell+1}$ splits the part $G_{i,i+1}$ into two parts with boundaries $\{q_i, p, x\}$ and $\{q_{i+1}, p, x\}$. By corollary 3, both these parts are not small. Thus, $\mathrm{Part}(\mathfrak{S}_{\ell+1})$ contains no small parts and by theorem 1 this set generates a flower. The induction step is proved. $\square$

Definition 5. We say that a flower F *contains* a flower F', if they have common center and $V(F') \subset V(F)$. We call a flower F *maximal*, if it is not contained in another flower.

Lemma 4. *Let $F = (p; q_1, \dots, q_m)$ be a maximal flower. Then the following statements hold.*

1) *There is no set $T \in \mathfrak{R}_3(G) \setminus \mathfrak{R}(F)$, which contains p and is dependent with at least one set of the flower F.*

2) *For each vertex $v \in \mathrm{Int}(G_{i,i+1})$ there exists a path between q_i and q_{i+1}, which does not pass through v, and all inner vertices of this path lie in* $\mathrm{Int}(G_{i,i+1})$.

Proof. 1) Assume the contrary. Then the set S, dependent with a set of the flower F, must be dependent with some cutset of $\mathfrak{R}(F)$. Hence, the dependence graph of the set $\mathfrak{S} = \mathfrak{R}(F) \cup \{S\}$ is connected. Moreover, every

cutset of the set $\mathfrak{S}$ contains a vertex p. Then by corollary 4 there exists such a flower F' that $\mathfrak{R}(F') \supset \mathfrak{S}$. Obviously, F' contains F and these flowers are different. A contradiction with maximality of F.

2) Assume the contrary. Then it is easy to see, that the set $T = \{v, p, q_{i+2}\}$ separates q_i from q_{i+1}, i.e., T is a cutset dependent with $Q_{i,i+1}$. A contradiction with item 1. □

Remark 3. 1) It is easy to reconstruct the center and petals of a flower F by the set $\mathfrak{R}(F)$.

2) Obviously, a flower F contains a flower F' if and only if $\mathfrak{R}(F) \subset \mathfrak{R}(F')$.

3) It is easy to derive from item 1 of lemma 4 that every flower is contained in unique maximal flower.

2.2 Vertex-edge cuts

Definition 6. 1) Let a *cutting set* be any set $M \in V(G) \cup E(G)$, for which the graph $G - M$ is disconnected. Obviously, every cutting set of a triconnected graph contains at least three elements.

Denote by $\mathfrak{M}_i(G)$ (where $i \in \{0, 1, 2, 3\}$) the set, consisting of all cutting sets with i edges and $3 - i$ vertices of the graph G. Let $\mathfrak{M}(G) = \cup_{i=1}^{3} \mathfrak{M}_i(G)$, $\mathfrak{M}^+(G) = \mathfrak{M}(G) \cup \mathfrak{M}_0(G)$. Note, that $\mathfrak{M}_0(G) = \mathfrak{R}_3(G)$.

All cutting sets of $\mathfrak{M}(G)$ we call *vertex-edge cuts*, or simply *cuts*.

2) Let $M, N \in \mathfrak{M}^+(G)$. If N contains all vertices of M and for every edge $e \in M$ the set N contains either e, or an end of e, we say that M *contains* N (or N *is contained* in M).

If a cut $M \in \mathfrak{M}(G)$ is not contained in any other cut of $\mathfrak{M}(G)$, we call it a *maximal* cut.

We say that a cutting set $M \in \mathfrak{M}^+(G)$ can be *complemented* by an edge ab (or an edge ab *complements* M), if an end of ab (let it be a) belongs to M and after changing in M the vertex a to the edge ab we obtain a cut from $\mathfrak{M}(G)$.

Note, that a cut is maximal if and only if it cannot be complemented by an edge.

Remark 4. Let $M \in \mathfrak{M}(G)$.

1) There is no vertex of M, which is incident to an edge of M.

2) If two edges of the cut M have common end x, then $\{x\}$ is a connected component of the graph $G - M$. Indeed, otherwise after replacing these two edges by a vertex x we obtain a cutting set of two elements, that is impossible in triconnected graph G.

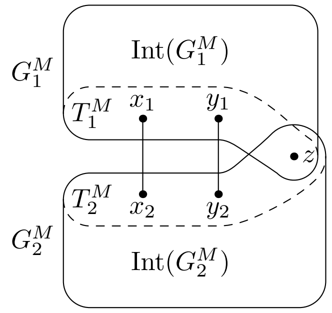

Figure 2: Decomposition of the graph by a cut from $\mathfrak{M}_2(G)$

3) We can consider the cut M as a subgraph of G (containing vertices of M, edges of M and ends of these edges). We denote by $V(M)$ the vertex set of this subgraph.

Let a cut $M \in \mathfrak{M}(G)$ contain an edge x_1x_2. It is clear, that the graph $G-M$ has exactly two connected components: one of them contains x_1, and the other contains x_2.

Definition 7. Let $x_1x_2 \in M \in \mathfrak{M}(G)$ and the graph $G-M$ have two connected components H_1 and H_2, such that $x_1 \in H_1$ and $x_2 \in H_2$. Then we set the notations $G_i^M = V(G) \setminus H_{3-i}$ and $T_i^M = G_i^M \cap V(M)$ for $i \in \{1,2\}$.

We call the sets G_1^M and G_2^M *parts of M-decomposition* and use the notation $\mathrm{Part}(M) = \{G_1^M, G_2^M\}$. We call *interior* of the part G_i^M the set $\mathrm{Int}(G_i^M) = G_i^M \setminus T_i^M$. We call *neighborhood* of the part G_i^M the set $\mathrm{Nb}(G_i^M) = G_i^M \cup V(M)$. Here $i \in \{1,2\}$. We call *boundaries* of the cut M the sets T_1^M and T_2^M.

For any cut $M \in \mathfrak{M}(G)$ we shall use these notations. Every edge of the cut M we shall write such that it first end lies in G_1^M, and second end lies in G_2^M.

The sets T_i^M, G_i^M and $\mathrm{Int}(G_i^M)$ are shown on fig. 2 for a cut $M \in \mathfrak{M}_2(G)$.

Remark 5. 1) Note, that the set G_i^M is obtained from the set H_i by adding of all vertices (but not edges!) of the cut M. Thus, the part G_i^M contains all vertices of the cut M and exactly one end of each edge of the cut M. The set T_i^M also contains all vertices of the cut M and exactly one end of each edge of the cut M, and does not contain other vertices. Hence, if edges of M have no common ends, then $|T_1^M| = |T_2^M| = 3$. If edges of M have common ends, then by remark 4 one connected component of the graph $G-M$ (denote it by H_1) consists of a single vertex. In this case $|T_1^M| = 1$ and $|T_2^M| = 3$.

2) Note also, that the definition of a part of M-decomposition is compatible with the definition of a part of decomposition of the graph by a cutset: G_1^M

and G_2^M are maximal (with respect to inclusion) sets not splitted by M.

3) Note, that a part of M-decomposition, by contrast of a part of decomposition by a cutset, can consist of a single vertex. It happens when a cut M consists of three edges, incident to a vertex of degree 3.

Lemma 5. *Let $x \in M \in \mathfrak{M}^+(G)$, $xy \in E(G)$ and H be a connected component of the graph $G - M$ containing y. Then the cutting set M can be complemented by an edge xy if and only if y is the only vertex of the component H which is adjacent to x.*

Proof. Let M' be a set obtained from M by replacing a vertex x by an edge xy. If the graph $G - M'$ is disconnected, then it consists of exactly two connected components, one of them contains the vertex x, and the other component contains y. Note, that all vertices of the component H lie in the same connected component of the graph $G - M'$. Thus, if the vertex x is adjacent in the graph $G - M'$ to a vertex of H, then the vertices x and y are connected in $G - M'$, i.e. the graph $G - M'$ is connected.

On the other side, if the vertex x is not adjacent to any vertex of the component H, except y, then there is no path between x and y in the graph $G - M'$, i.e. this graph is disconnected. $\square$

Corollary 5. *If $M \in \mathfrak{M}_2(G)$, then there exists not more than one edge which complement M.*

Proof. Let x be the only vertex of the cut M and H_1, H_2 be connected components of the graph $G - M$. Then by lemma 5 there is not more than one edge from x to each of these components by which M can be complemented. Obviously, $V(G) = H_1 \cup H_2 \cup \{x\}$. Since the graph G is triconnected, $d(x) \geq 3$. Thus the vertex x cannot be adjacent to exactly one vertex of the component H_1 and exactly one vertex of the component H_2. Hence, the cut M can be complemented by not more than one edge. $\square$

Corollary 6. *Let cutsets $S, T \in \mathfrak{R}_3(G)$ be dependent and $\{x, y\} \in \mathrm{Part}(\{S, T\})$. Then each of the sets S and T can be complemented by the edge xy.*

Proof. By corollary 3 we have $S \cap T = \varnothing$. Without loss of generality we may suppose, that $x \in S$, $y \in T$. Let $x \in F \in \mathrm{Part}(T)$, $y \in H \in \mathrm{Part}(S)$. By corollary 2 the vertices x and y are adjacent and y is the only vertex of the part H adjacent to x. Then by lemma 5 the set S can be complemented by the edge xy. Similarly, the set T can be complemented by the edge xy. $\square$

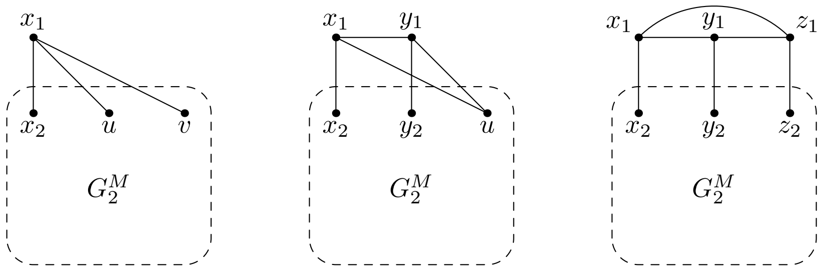

Figure 3: Degenerate cuts.

Definition 8. We call a cut $M \in \mathfrak{M}(G)$ *nondegenerate*, if $\mathrm{Int}(G_1^M) \neq \varnothing$ and $\mathrm{Int}(G_2^M) \neq \varnothing$. Otherwise, we call this cut *degenerate.*

We call a cutting set $M \in \mathfrak{M}^+(G)$ *trivial*, if one connected component of the graph $G - M$ consists of a single vertex, and *nontrivial* otherwise.

A case $\mathrm{Int}(G_1^M) = \mathrm{Int}(G_2^M) = \varnothing$ is not interesting, because in this case the graph G contains not more than 6 vertices. Further we suppose, that for every degenerate cut M exactly one of the sets $\mathrm{Int}(G_1^M)$ and $\mathrm{Int}(G_2^M)$ is empty.

Remark 6. 1) A degenerate cut containing exactly one edge is trivial. Indeed, if a cut $M = \{u, v, x_1x_2\}$ is degenerate ($\mathrm{Int}(G_1^M) = \varnothing$), then since the graph G is triconnected, the vertex x_1 is adjacent to the vertices x_2, u, v and only them. Thus the set $T_2^M = \{x_2, u, v\}$ separates the vertex x_1 from other vertices of the graph. Sets $\{x_1u, x_1v, x_1x_2\}$ and $\{u, x_1v, x_1x_2\}$ also are cuts. By remark 4, if a cut contains adjacent edges, it is trivial.

2) The structure of a degenerate nontrivial cut also can be simply described. It follows from written above, that such cut contains more than one edge.

If a cut $M = \{u, x_1x_2, y_1y_2\}$ is degenerate ($\mathrm{Int}(G_1^M) = \varnothing$) and nontrivial, then the vertex x_1 is adjacent to the vertices y_1, x_2, u and only them, and the vertex y_1 is adjacent to x_1, y_2, u and only them.

If a cut $M = \{x_1x_2, y_1y_2, z_1z_2\}$ is degenerate and nontrivial (again $\mathrm{Int}(G_1^M) = \varnothing$), then the vertices x_1, y_1, z_1 are pairwise adjacent and except this edges the vertices x_1, y_1, z_1 are incident to edges of the cut M and only them.

Degenerate cuts with one, two and three edges are shown on figure 3.

Lemma 6. *Let $M \in \mathfrak{M}(G)$ be a nontrivial cut. Then the following statements hold.*

1) *Every set, which contains all vertices of M and exactly one end of each edge of M and differs from T_1^M and T_2^M, is a cutset. Moreover, this*

cutset splits the graph into two parts, one of which contains G_1^M, and the other contains G_2^M.

2) *If* $\mathrm{Int}(G_2^M) \neq \varnothing$*, then* T_2^M *is a cutset. Moreover,* $\mathrm{Nb}(G_1^M) \in \mathrm{Part}(T_2^M)$ *and* G_2^M *is a union of several parts of* $\mathrm{Part}(T_2^M)$*. If the cut* M *is nondegenerate, then both* T_1^M *and* T_2^M *are cutsets.*

Proof. 1) Let R be any such set. Since R does not coincide with T_1^M and T_2^M, then the sets $G_1^M \setminus R$ and $G_2^M \setminus R$ are nonempty. Obviously, R separates these two sets from each other, thus, R is a cutset.

Let us prove, that vertices of the set $G_1^M \setminus R$ are connected in the graph $G - R$. Indeed, let $x_1 \in T_1^M \setminus R$. Then the cut M contains an edge x_1x_2 and $x_2 \in R$. Since the cut M is nontrivial, then x_1 is the only vertex outside the part G_2^M which is adjacent to x_2. However, each connected component of the graph $G - R$ must contain a vertex adjacent to x_2. Since any connected component of $G - R$ which contains a vertex of $G_1^M \setminus R$ contain no vertices of G_2^M, then all vertices of the set $G_1^M \setminus R$ are in the same connected component of the graph $G - R$.

Analogously, all vertices of the set $G_2^M \setminus R$ are in the same connected component of the graph $G - R$. Hence there are exactly two connected components in the graph $G - R$.

2) It is easy to see, that the set T_2^M separates G_2^M from $\mathrm{Nb}(G_1^M)$. Hence, T_2^M is a cutset. The fact, that vertices of the set $G_1^M \setminus T_2^M$ are connected in the graph $G - R$, is proved as well as in item 1. It is clear, that a part of $\mathrm{Part}(T_2^M)$ containing $G_1^M \setminus T_2^M$ is $\mathrm{Nb}(G_1^M)$. $\square$

Corollary 7. *Let* $M \in \mathfrak{M}(G)$*, a cutset* $S \in \mathfrak{R}_3(G)$ *be dependent with* T_2^M*,* $S \cap G_2^M = \{x\}$ *and* S *separates a vertex* $y \in T_2^M$ *from other vertices of* T_2^M*. Then the cut* M *can be complemented by the edge* xy*.*

Proof. Let $x \in H \in \mathrm{Part}(T_2^M)$ (it is easy to check, that $H = G_2^M$) and $y \in F \in \mathrm{Part}(S)$. By condition, $S \cap H = \{x\}$ and $T_2^M \cap F = \{y\}$. Then by corollary 2, we have that $\{x, y\} \in \mathrm{Part}(\{S, T_2^M\})$, the vertices x and y are adjacent and x is the only vertex of the part G_2^M adjacent to y. Hence, by lemma 5 the edge xy complements the cut M. $\square$

Remark 7. Obviously, all cutsets of lemma 6 are contained in the cut M. Moreover, all these sets, except T_1^M and T_2^M, are pairwise dependent. The cutsets T_1^M and T_2^M are independent with each other and with all other cutsets, contained in M.

Definition 9. The cutsets described in item 1 of lemma 6 we call *inner sets* of the cut M. The set consisting of all inner sets of M we denote by $\mathfrak{R}(M)$.

2.2.1 Singular edges

Lemma 7. *Let cuts $M_1, M_2 \in \mathfrak{M}_2(G)$ have two common edges. Then there is a cut $M \in \mathfrak{M}_3(G)$ containing both M_1 and M_2.*

Proof. Let $M_1 = \{x_1x_2, y_1y_2, z_1\}$ and z_2 be the only vertex of M_2. Without loss of generality we may suppose, that $z_2 \in G_2^{M_1}$. Then M_2 does not split $G_1^{M_1}$. In particular, M_2 does not split $\{x_1, y_1, z_1\}$. Thus we may suppose, that $M_2 = \{x_1x_2, y_1y_2, z_2\}$ and $z_1 \in G_1^{M_2}$.

Note, that then the cutset $T_2^{M_1} = \{x_2, y_2, z_1\}$ separates z_2 from $\{x_1, y_1\}$, and the cutset $T_1^{M_2} = \{x_1, y_1, z_2\}$ separates z_1 from $\{x_2, y_2\}$. Hence, the cutsets $T_2^{M_1}$ and $T_1^{M_2}$ are dependent and by corollary 7 we can complement the cut M_1 by the edge z_1z_2 and obtain desired cut $M = \{x_1x_2, y_1y_2, z_1z_2\}$. $\square$

Corollary 8. *Two maximal cuts cannot have more than one common edge.*

Proof. Let cuts $M_1, M_2 \in \mathfrak{M}(G)$ have two common edges. If $M_i \in \mathfrak{M}_2(G)$, then we set $M_i' = M_i$, else (when $M_i \in \mathfrak{M}_3(G)$), we obtain M_i' from M_i replacing the edge which is not in M_{3-i} by one of its ends. Clearly, we can perform this replacement such that cuts M_1' and M_2' would be different. Then by lemma 7 there is a cut $M \in \mathfrak{M}_3(G)$, containing both cuts M_1' and M_2'. However, by corollary 5 each of the cuts M_1' and M_2' can be contained in not more than one cut from $\mathfrak{M}_3(G)$. Hence, each of the cuts M_1 and M_2 is contained in M or coincide with M, i.e. at least one of the cuts M_1 and M_2 is not maximal. $\square$

Definition 10. We call an edge $e \in E(G)$ *singular*, if there exist different vertices $u, v, t, w \in V(G)$ such that $\{u, v, e\}, \{t, w, e\} \in \mathfrak{M}(G)$.

Remark 8. Let $\{a_1a_2, b_1b_2, c_1c_2\} \in \mathfrak{M}_3(G)$. It is easy to see, that all edges a_1a_2, b_1b_2, c_1c_2 are singular.

Definition 11. Let $M = \{u, v, x_1x_2\}$, $N = \{t, w, x_1x_2\} \in \mathfrak{M}_1(G)$, and the vertices u, v, t, w are different. We call the cuts M and N *independent*, if one of the parts G_1^M and G_1^N contains the other. Otherwise, we call these cuts *dependent*.

Clearly, the cuts M and N are independent if and only if either $t, w \in \mathrm{Int}(G_1^M)$, or $t, w \in \mathrm{Int}(G_2^M)$.

Lemma 8. *Let cuts $M, N \in \mathfrak{M}_1(G)$ be dependent and both contain an edge x_1x_2. Then there exists a cut $\{x_1x_2, y_1y_2, z_1z_2\} \in \mathfrak{M}_3(G)$ such that $M = \{x_1x_2, y_1, z_2\}$, $N = \{x_1x_2, y_2, z_1\}$.*

Proof. Let $M \cap G_1^N = \{y_1\}$, $M \cap G_2^N = \{z_2\}$, $N \cap G_1^M = \{z_1\}$, $N \cap G_2^M = \{y_2\}$. Consider sets $T_2^M = \{x_2, y_1, z_2\}$ and $T_1^N = \{x_1, y_2, z_1\}$. Note, that T_2^M separates the vertex y_2 from $\{x_1, z_1\}$, and T_1^N separates the vertex y_1 from $\{x_2, z_2\}$. Thus, by corollary 7 both cuts M and N can be complemented by the edge y_1y_2. Obtained cuts $\{x_1x_2, y_1y_2, z_1\}$ and $\{x_1x_2, y_1y_2, z_2\}$ by lemma 7 can be complemented by the edge $\{z_1z_2\}$. As a result, we obtain desired cut $\{x_1x_2, y_1y_2, z_1z_2\}$. □

Theorem 2. *For vertices $x_1, x_2 \in V(G)$ two following statements are equivalent.*

1° *Vertices x_1 and x_2 are adjacent, x_1x_2 is a singular edge.*

2° *There exist dependent cutsets $S, T \in \mathfrak{R}_3(G)$ such that $x_1 \in S$, $x_2 \in T$ and $\{x_1, x_2\} \in \mathrm{Part}(\{S, T\})$.*

Proof. 2° $\Rightarrow$ 1°. Let the condition 2° hold. Then, by corollary 6 both cutsets S and T can be complemented by the edge x_1x_2. Hence, the edge x_1x_2 is singular.

1° $\Rightarrow$ 2°. If the condition 1° holds, then there exist different vertices $u, v, t, w \in V(G)$ such that $M = \{u, v, x_1x_2\}, N = \{t, w, x_1x_2\} \in \mathfrak{M}(G)$. Consider two cases.

a. Suppose, that M *and* N *are independent.* Without loss of generality we may assume, that $G_1^M \supset G_1^N$ and $G_2^M \subset G_2^N$. Consider disjoint sets T_1^M and T_2^N. In our case $t, w \in \mathrm{Int}(G_1^M)$, $x_2 \notin G_1^M$. Then by lemma 6 the set T_1^M is a cutset and separates x_2 from $\{t, w\}$. Similarly, T_2^N is a cutset and separates x_1 from $\{u, v\}$. Hence, the cutsets $T_1^M = \{x_1, u, v\}$ and $T_2^N = \{x_2, t, w\}$ are dependent and by corollary 2 we have $\{x_1, x_2\} \in \mathrm{Part}(\{T_1^M, T_2^N\})$.

b. Suppose, that M *and* N *are dependent.* Then by lemma 8 there exists a cut $\{x_1x_2, y_1y_2, z_1z_2\} \in \mathfrak{M}_3(G)$ such that $M = \{x_1x_2, y_1, z_2\}$, $N = \{x_1x_2, y_2, z_1\}$. Consider sets $S = \{x_1, y_2, z_2\}$ and $T = \{x_2, y_1, z_1\}$. By lemma 6 we have $S, T \in \mathfrak{R}_3(G)$, moreover, S separates the vertex x_2 from y_1, z_1, and T separates the vertex x_1 from $\{y_2, z_2\}$. Thus, the sets S and T are dependent and by corollary 2 we have $\{x_1, x_2\} \in \mathrm{Part}(\{S, T\})$. □

2.3 Trivial cutsets and triple cuts

Remind, that a cut M is called *trivial*, if one connected component of the graph $G - M$ consists of one vertex. Obviously, degree of this vertex is 3. In this section we study trivial cutsets.

Definition 12. Let $T \in \mathfrak{R}_3(G)$ be a trivial cutset, separating a vertex a from other vertices. Let a set $S \in \mathfrak{R}_3(G)$, containing the vertex a, be such that $|\mathrm{Part}(S)| = 3$ and interior of every part of $\mathrm{Part}(S)$ contains a vertex of

the set T. Then we say, that the trivial set T is *subordinated* to the set S. Denote by $\mathfrak{D}$ the set of all 3-cutsets, which have a subordinated trivial cutset.

Remark 9. It is clear by definition, that if $S \in \mathfrak{D}$, then there exists a vertex $a \in S$ of degree 3.

Lemma 9. 1) *If a cutset $S \in \mathfrak{R}_3(G)$ splits G into more than three parts, then S is independent with all cutsets of $\mathfrak{R}_3(G)$. If a cutset $S \in \mathfrak{R}_3(G)$ splits G into three parts and S is dependent with $T \in \mathfrak{R}_3(G)$, then the cutset T is trivial and subordinated to S.*

2) *A trivial cutset can be subordinated to not more than one cutset.*

Proof. 1) Let cutsets $S, T \in \mathfrak{R}_3(G)$ be dependent. Then by corollary 1 we have $|\mathrm{Part}(S)| \leq 3$ and $|\mathrm{Part}(T)| \leq 3$. Let $\mathrm{Part}(S) = \{H_1, H_2, H_3\}$. If $|\mathrm{Part}(T)| = 3$, then, by remark 1 we have $|V(G)| = 6$, this case is not interesting for us. It is enough to consider the case $|\mathrm{Part}(T)| = 2$. Let $\mathrm{Part}(T) = \{F_1, F_2\}$. We enumerate the parts such that $\mathrm{Int}(F_1) \cap S = \{a\}$, $|\mathrm{Int}(F_2) \cap S| = 2$. Then by corollary 2 the set S splits F_1 into three empty parts. Hence, $\mathrm{Int}(F_1) = \{a\}$, i.e. T is a trivial cutset subordinated to S.

2) Let T be a trivial cutset subordinated to both cutsets S and S'. By item 1 then $|\mathrm{Part}(S)| = |\mathrm{Part}(S')| = 3$, hence, the cutsets S and S' are independent. Let $A \in \mathrm{Part}(S)$ be a part containing S'. Two vertices of the cutset T lie outside A. Clearly, S' cannot separate these two vertices from each other. We have a contradiction. □

Further in this section we consider a cutset $S \in \mathfrak{D}$.

Definition 13. Consider a vertex $a \in S$ of degree 3. Three vertices adjacent to a form a cutset, separating a from other vertices. We denote this cutset by $T(a)$ and call it *neighborhood* of the vertex a.

It easy to see, that if $a \in S$ and $d(a) = 3$, then $T(a)$ is a trivial cutset, subordinated to S. Moreover, the set S can be complemented by any edge aa_i where $a_i \in T(a)$.

Let $\mathrm{Part}(S) = \{A_1, A_2, A_3\}$. We replace in S every vertex a of degree 3 by an edge connecting a with the vertex of $T(a) \cap \mathrm{Int}(A_i)$ and denote obtained cut by M_i. It follows from above, that $M_1, M_2, M_3 \in \mathfrak{M}(G)$.

The cuts M_1, M_2, M_3 may be not maximal. If M_i is contained in a cut from $\mathfrak{M}_3(G)$, denote this cut by M'_i (obviously, this cut is unique). In all other cases (among them the case when $M_i \in \mathfrak{M}_1(G)$ is contained in a cut from $\mathfrak{M}_2(G)$) we set $M'_i = M_i$.

Obviously, $V(M_i) \subset V(M'_i) \subset A_i$. Moreover, the set S is a bound of both cuts M_i and M'_i. By lemma 6 we have, that $A_{i+1} \cup A_{i+2} \in \mathrm{Part}(M_i)$

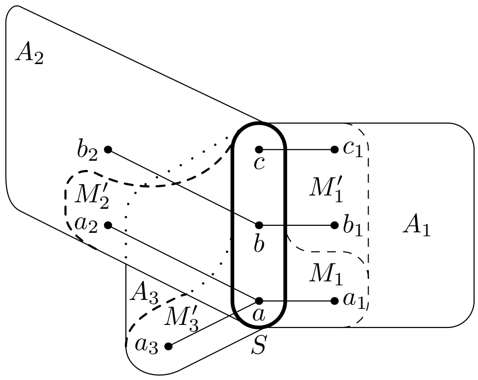

Figure 4: A triple cut with one trivial cutset

and $A_{i+1} \cup A_{i+2} \in \mathrm{Part}(M_i')$ (the numeration is cyclic modulo 3). Denote the *other part* of $\mathrm{Part}(M_i)$ by B_i, and its boundary by T_i. We denote by B_i' the part of $\mathrm{Part}(M_i')$, contained in B_i, and its boundary denote by T_i'. The neighborhood of B_i as a part of $\mathrm{Part}(M_i)$ and the neighborhood of B_i' as a part of $\mathrm{Part}(M_i')$ are defined. It is easy to see, that $\mathrm{Nb}(B_i) = \mathrm{Nb}(B_i') = A_i$.

Note, that the cut M_i (and, consequently, the cut M_i') can be trivial. In this case $|B_i'| = 1$. Moreover, if in this case all vertices of the cutset S are of degree 3, then also $|B_i| = 1$.

Definition 14. We call $F = M_1 \cup M_2 \cup M_3$ a *triple cut*, and $\mathrm{Nb}(F) = V(M_1') \cup V(M_2') \cup V(M_3')$ — its *neighborhood*. The set S we call a *line* of triple cut. All inner cutsets of the cuts M_i, the set S and all cutsets subordinated to S we call *inner sets* of this triple cut. We call the sets T_1, T_2, T_3 *boundaries* of our triple cut and the sets T_1', T_2', T_3' — boundaries of its neighborhood. Set $V(F) = V(M_1) \cup V(M_2) \cup V(M_3)$ and $\mathrm{Part}(F) = \{B_1, B_2, B_3\}$.

An example of a triple cut is shown on figure 4. In this example the line $S = \{a, b, c\}$ of this triple cut has one subordinated cutset $T(a) = \{a_1, a_2, a_3\}$. Here $M_1 = \{aa_1, b, c\}$, $M_1' = \{aa_1, bb_1, cc_1\}$, $M_2 = M_2' = \{aa_2, b, c\}$, $M_3 = M_3' = \{aa_3, b, c\}$. Note, that the cut M_2' coincides with M_2, in spite of it is not maximal: the cut M_2 can be complemented by the edge bb_2.

Remark 10. A nontrivial definition of the cut M_i' is concerned with our aim. On one side, we want to give the most simple description of the parts of decomposition of the graph by a set of cutsets, contained in the neighborhood of a triple cut, which are not its boundaries. On the other side we want each 3-cutset not contained in $\mathrm{Nb}(F)$ to separate from the neighborhood not more than one vertex. Later we shall show, that our definition of the neighborhood satisfies both these conditions.

3 Further properties of basic structures

The structures described in previous section are generated by sets of dependent cutsets. In this section we describe their connection with each other and with other cutsets.

3.1 Inner cutsets

Lemma 10. *Let S be a set of three petals of a flower F. Then S is not a cutset.*

Proof. Let $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$. Then all vertices of the set $G_{i,i+1} \setminus S$ are connected in $G - S$, and p is among them. Thus all vertices of $G - S$ are connected, except, may be, some petals of the flower, not belonging to nonempty parts of $\mathrm{Part}(F)$. But any such petal q_j belongs to two empty parts of $\mathrm{Part}(F)$ and, by remark 2, is adjacent to p. Thus, the graph $G - S$ is connected. □

Corollary 9. *Any 3-cutset, which is a subset of the vertex set of a flower, contains its center, i.e. it is either an inner set of this flower, or its boundary.*

Lemma 11. *Let $M \in \mathfrak{M}(G)$ be a nontrivial cut, containing an edge x_1x_2. Then there is no cutset $S \in \mathfrak{R}_3(G)$, which contains both x_1 and x_2.*

Proof. Suppose the converse, let $x_1, x_2 \in S \in \mathfrak{R}_3(G)$. Without loss of generality suppose, that $S = \{x_1, x_2, t\}$, where $t \in G_1^M$. Since $\mathrm{Int}(G_2^M)$ is a union of interiors of several parts of $\mathrm{Part}(T_2^M)$ and $S \cap \mathrm{Int}(T_2^M) = \varnothing$, then by lemma 2 all vertices of the set $G_2^M \setminus S$ are connected in the graph $G - S$. Moreover, they are also connected with vertices of the set $T_1^M \setminus S$, because each vertex of this set either belongs to the set $T_2^M \setminus S$, or is adjacent to a vertex of this set.

Consider any vertex $w \in \mathrm{Int}(G_1^M) \setminus S$, if this set is nonempty. Since $\mathrm{Int}(G_1^M)$ is a union of interiors of several parts of $\mathrm{Part}(T_1^M)$, by Menger's theorem there exist three vertex-disjoint paths inside G_1^M from w to three vertices of the set T_1^M. Since $|S \cap G_1^M| = 2$, at least one of these paths omit S. Hence the vertex w is connected in the graph $G - S$ with the set $G_2^M \setminus S$. That means the graph $G - S$ is connected, we have a contradiction. □

Corollary 10. *Any 3-cutset, which is contained in the vertex set of a cut, contains all its vertices and exactly one vertex of each edge of this cut. Hence, this cutset is an inner set of this cut or its boundary.*

Lemma 12. *For a triple cut $F = M_1 \cup M_2 \cup M_3$ the following statements hold.*

1) *Any* 3*-cutset, which is contained in $V(F)$ is an inner set of this triple cut or its boundary.*

2) *Any* 3*-cutset which is contained in the neighborhood of a triple cut F either is subordinated to the line of F, or is contained in one of the cuts M'_1, M'_2, M'_3.*

Proof. 1) A cutset dependent with the line of a triple cut F by lemma 9 is subordinated to it. Hence, this cutset is an inner set of F. The cutset, independent with the line of F is contained in one of the sets $V(M_i)$ (where $i \in \{1, 2, 3\}$). Then by corollary 10 this cutset is either an inner set or a boundary of the cut M_i, and by definition it is an inner set or a boundary of the triple cut F.

2) Similarly to item 1, a set independent with the line of F is contained in one of the sets $V(M'_i)$ and, consequently, is contained in the cut M'_i. □

3.2 Connection between flowers and cuts

In this section we consider a question, in what cases vertex sets of a cut and a flower coincide, or one of them is a subset of the other.

Definition 15. We say that a flower F *is contained* in a cut M, if $V(F) \subset V(M)$. We say that a cut M *is contained* in a flower F, if $V(M) \subset V(F)$.

Lemma 13. *A flower contained in a cut has exactly* 4 *petals and two non-neighboring empty parts.*

Proof. Let M be a cut and F be a flower such that $V(F) \subset V(M)$. Obviously, $|V(F)| \le |V(M)| \le 6$. Moreover, if $V(F) = 6$, then $M \in \mathfrak{M}_3(G)$, $V(F) = V(M)$ and cut M is nontrivial. Then the center of F is an end of an edge of the cut M, and the other end of this edge is a petal of F. Hence, both ends of this edge belong to a cutset of F. That is impossible by lemma 11.

Hence, $V(F) = 5$ and the flower F has exactly 4 petals. Similarly to written above, the center of F cannot be connected with its petal by an edge of the cut M. Thus, petals of F form two pairs of vertices, connected by edges of M. Obviously, the petals, which are ends of an edge of M are neighboring, because non-neighboring petals are not adjacent. Let them be q_1 and q_2. Then the part $G_{1,2}$ is empty, since otherwise q_1 and q_2 are connected by a path inside $G_{1,2}$, and this path does not contain edges of M. Clearly, this is impossible. □

Therefore, only a flower with 4 petals can be contained in a cut. It is easy to see, that for every nontrivial cut of $\mathfrak{M}_2(G)$ its two inner sets generate a flower with 4 petals, which vertex set coincide with the vertex set of considered cut. This is the only case, when vertex sets of a cut and of a flower coincide. For every nontrivial cut $M \in \mathfrak{M}_3(G)$ there are 6 cuts of $\mathfrak{M}_2(G)$ contained in M and 6 flowers correspondent to these cuts. These 6 flowers are contained in M.

Definition 16. We call a flower F *nondegenerate*, if it is not contained in any cut $M \in \mathfrak{M}_3(G)$, and *degenerate* otherwise.

Now consider more often situation, when a flower contains a cut.

Lemma 14. *Let $F = (p; q_1, \dots, q_m)$ be a maximal flower and $\{q_i, p, q_j x\} \in \mathfrak{M}(G)$. Then one of the following two statements holds.*

1° *The vertex x is a petal of F, neighboring with q_j, and $\{q_j, p, x\}$ is an empty part of* $\mathrm{Part}(F)$.

2° *The conditions $\{i, j\} = \{k, k+1\}$ and $x \in \mathrm{Int}(G_{k,k+1})$ hold. Moreover, if $|\mathrm{Part}(Q_{k,k+1})| = 2$, then the vertices q_k and q_{k+1} are adjacent.*

Proof. Clearly, both ends of the edge $q_j x$ lie in the same part of $\mathrm{Part}(F)$. Without loss of generality we may suppose that $x \in G_{j,j+1}$. Let $x \in H \in \mathrm{Part}(Q_{j,i})$. Note, that if $|\mathrm{Part}(Q_{j,i})| = 2$, then $H = G_{j,i}$ (this condition for $i \neq j+1$ certainly holds). By lemma 5 we have, that q_j is not adjacent to different from x vertices of $\mathrm{Int}(H)$.

Let $x \neq q_{j+1}$. If $i \neq j+1$, then it is easy to see, that the set $T = \{q_i, p, x\}$ separates q_j from q_{j+1} and, consequently, T is a cutset. By lemma 4, this contradicts the maximality of the flower F. Thus, $i = j+1$. Now note, that if $|\mathrm{Part}(Q_{j,j+1})| = 2$, then q_j is not adjacent to vertices of $\mathrm{Int}(G_{j,j+1})$ different from x. If q_j and q_{j+1} are not adjacent, then, clearly, the set $T_1 = \{q_{j+2}, p, x\}$ separates q_j from q_{j+1} and, consequently, T_1 is a cutset. That also contradicts maximality of the flower F. Hence, in this case the vertices q_j and q_{j+1} are adjacent.

Let $x = q_{j+1}$. Then q_j is not adjacent to any vertex of $\mathrm{Int}(G_{j,j+1})$, whence it follows that the part $G_{j,j+1}$ is empty. $\square$

Corollary 11. 1) *Let M be a nontrivial cut and $F = (p; q_1, \dots, q_m)$ be a flower such that $V(M) \subset V(F)$. Then $p \in M$ and each edge of the cut M takes the form $q_j q_{j+1}$, where $G_{j,j+1}$ is an empty part.*

2) *Let $F = (p; q_1, \dots, q_m)$ be a flower and $\mathrm{Int}(G_{i,i+1}) = \varnothing$. Then $\{q_j, p, q_i q_{i+1}\} \in \mathfrak{M}_1(G)$ for every $j \notin \{i, i+1\}$. Moreover, if $\mathrm{Int}(G_{j,j+1}) = \varnothing$ and $i \neq j$, then $\{q_j q_{j+1}, p, q_i q_{i+1}\} \in \mathfrak{M}_2(G)$.*

Proof. 1) Since at least one boundary of the cut M by lemma 6 is a cutset, by lemma 10 this boundary contains the center of the flower F. Consequently, $p \in V(M)$. Suppose that $px \in M$. Clearly, there exist a cutset of flower F containing both p and x, that contradicts lemma 11. Hence, $p \in M$.

Let $xy \in M$. Consider a maximal flower F' which contains F. Since x and y are petals of F, they are also petals of F'. By lemma 14, the petals x and y must be neighboring in the flower F' (hence, in the flower F too), and correspondent to x and y part of $\mathrm{Part}(F)$ is empty.

2) It is easy to verify, that each of these sets separates q_i from q_{i+1}. □

Remark 11. A trivial cut can be contained in a flower, if this flower has two neighboring empty parts. Then edges connecting their common petal with neighboring petals and with the center of flower form a trivial cut. It is easy to check, that any trivial cut, contained in a flower is of this type.

Definition 17. Let $G_{i,i+1} \in \mathrm{Part}(F)$. Define a set $M_{i,i+1}$ as follows.

1° $p \in M_{i,i+1}$;

2° $q_{i-1}q_i \in M_{i,i+1}$, if $\mathrm{Int}(G_{i-1,i}) = \varnothing$, and $q_i \in M_{i,i+1}$ otherwise;

3° $q_{i+1}q_{i+2} \in M_{i,i+1}$, if $\mathrm{Int}(G_{i+1,i+2}) = \varnothing$, and $q_{i+1} \in M_{i,i+1}$ otherwise.

If at least one of the parts $G_{i-1,i}$ and $G_{i+1,i+2}$ is empty, we call $M_{i,i+1}$ a *boundary cut* of the part $G_{i,i+1}$.

Remark 12. 1) The fact that $M_{i,i+1} \in \mathfrak{M}^+(G)$ obviously follows from corollary 11.

2) Note, that a boundary cut can be not maximal. If x is the only vertex of the part $G_{i,i+1}$ adjacent to p than the set $M_{i,i+1}$ can be complemented by an edge px.

3) Also note, that if $M_{i,i+1} \in \mathfrak{M}(G)$, then $G_{i,i+1} \in \mathrm{Part}(M_{i,i+1})$.

Lemma 15. *Let $F = (p; q_1, \dots, q_m)$ be a nondegenerate flower. Then the following statements hold.*

1) *If a cutset $Q_{i,i+1}$ can be complemented by an edge px, then $x \in \mathrm{Int}(G_{i,i+1})$.*

2) *If $M_{i,i+1} \in \mathfrak{M}(G)$, and the cut $M_{i,i+1}$ can be complemented by an edge, then this edge is px where $x \in \mathrm{Int}(G_{i,i+1})$ is the only vertex of the part $G_{i,i+1}$ adjacent to p.*

Proof. 1) Let $x \notin \mathrm{Int}(G_{i,i+1})$. Then $x \in V(G) \setminus G_{i,i+1}$. Note, that all vertices of the set $V(G) \setminus G_{i,i+1}$ are in the same connected component of the graph $G - Q_{i,i+1}$. By lemma 5 we have, that x is the only vertex of this component which is adjacent to p. From the other side the vertex p must be adjacent to at least one inner vertex of each nonempty part of $\mathrm{Part}(F)$ and, by remark 2, to a common petal of each two neighboring empty parts.

Then there is not more than one nonempty part among all different from $G_{i,i+1}$ parts. If all these parts are empty, then there are at least tree consecutive empty parts, i.e. there at least two petals not from $G_{i,i+1}$ adjacent to p, that is impossible. Thus, there are exactly two nonempty parts in $\mathrm{Part}(F)$ and no neighboring empty parts. That means $m = 4$, $\mathrm{Int}(G_{i-1,i}) = \mathrm{Int}(G_{i+1,i+2}) = \varnothing$ and $\mathrm{Int}(G_{i+2,i-1}) \neq \varnothing$. Then $M_{i,i+1} = \{q_i q_{i-1}, q_{i+1}q_{i+2}, p\}$, and, complementing it by an edge px, we obtain, that F is contained in the cut $M = \{q_i q_{i-1}, px, q_{i+1}q_{i+2}\} \in \mathfrak{M}_3(G)$, i.e. F is a degenerate flower. We obtain a contradiction.

2) Let the cut $M_{i,i+1}$ can be complemented by an edge e. Then the cutset $Q_{i,i+1}$ also can be complemented by this edge. Hence, by previous item, e cannot be an edge px, where $x \in \mathrm{Int}(G_{i+1,i})$.

Suppose, that $e = q_i v$ (case $e = q_{i+1}v$ is similar). Since $M_{i,i+1} \in \mathfrak{M}(G)$, we obtain, that $M_{i,i+1} = \{q_i, p, q_{i+1}q_{i+2}\}$. Consider two cases.

1. $v \in G_{i+1,i}$.

Then $v \in G_{i-1,i}$ and $\{q_{i+1}, p, q_i v\} \in \mathfrak{M}_1(G)$, hence, by lemma 14 we have $v = q_{i-1}$ and $\mathrm{Int}(G_{i-1,i}) = \varnothing$. But then $q_i v = q_i q_{i-1} \in M_{i,i+1}$.

2. $v \in G_{i,i+1}$.

In this case by lemma 6 we obtain, that $\{v, p, q_{i+2}\}$ is a cutset containing p and dependent with $Q_{i,i+1}$. Then by lemma 4 the flower F is not maximal, that contradicts the condition of lemma.

Thus, the only remained case is $e = px$ where $x \in \mathrm{Int}(G_{i,i+1})$. Then by lemma 5 we have, that x is the only vertex of the part $G_{i,i+1}$ adjacent to p. $\square$

Definition 18. If a cut $M_{i,i+1}$ can be complemented by an edge px where $x \in \mathrm{Int}(G_{i,i+1})$, then denote by $M^*_{i,i+1}$ the cut, obtained after complementing.

An example of a cut $M^*_{i,i+1}$ for the case when both parts $G_{i-1,i}$ and $G_{i+1,i+2}$ are empty is shown on figure 5.

3.3 Sets, splitting basic structures

In this section we consider 3-cutsets which split the vertex set of a cut, a triple cut, or a flower.

Lemma 16. *Let maximal nontrivial cut M and $S \in \mathfrak{R}_3(G) \setminus \mathfrak{R}(M)$ be such that S splits $V(M)$. Then $|\mathrm{Part}(S)| = 2$ and one of the following statements holds.*

1° *The cut M is contained in a flower, generated by the set $\mathfrak{S} = \mathfrak{R}(M) \cup \{S, T^M_1, T^M_2\}$.*

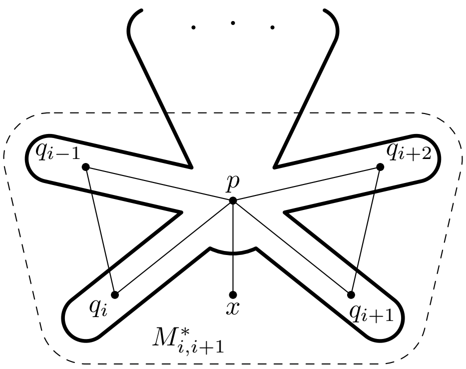

Figure 5: A cut $M^*_{i,i+1}$

2° *The set S is contained in a neighborhood of a part of* $\mathrm{Part}(M)$ *(let it be G_1^M). There exists such an edge $x_1x_2 \in M$ that S separates the vertex $x_1 \in T_1^M$ from other vertices of the set $V(M) \setminus S$, and $S \setminus G_1^M = \{x_2\}$.*

Proof. Consider two cases.

1. $S \cap \mathrm{Int}(G_1^M) \neq \varnothing$ and $S \cap \mathrm{Int}(G_2^M) \neq \varnothing$.

In this case the set S is dependent with all cutsets of $\mathfrak{R}(M)$ and also with T_1^M and T_2^M. Thus, the dependence graph of the set $\mathfrak{S}$ is connected.

Prove, that $|S \cap G_1^M| = 2$. Let $S \cap G_1^M = \{x\}$. Then $x \in \mathrm{Int}(G_1^M)$ and $S \cap T_1^M = \varnothing$. Whence by corollary 2 the set S must separate a single vertex of the set T_1^M (denote it by y) from other vertices of this set. But then by corollary 7 the cut M can be complemented by an edge xy. We obtain a contradiction with maximality of the cut M.

Thus, $|S \cap G_1^M| = 2$. Similarly, $|S \cap G_2^M| = 2$. Therefore, $S \cap G_1^M \cap G_2^M \neq \varnothing$. That is there exists a vertex $p \in M \cap S$. But this vertex belongs to all cutsets of the set $\mathfrak{S}$, whence by corollary 4 it follows, that the set $\mathfrak{S}$ generates a flower F, which contains the cut M. Since $S \in \mathfrak{R}(F)$, then $|\mathrm{Part}(S)| = 2$.

2. $S \cap \mathrm{Int}(G_2^M) = \varnothing$.

In this case $S \subset \mathrm{Nb}(G_1^M)$, the set S is independent with T_2^M and dependent with T_1^M. Similarly to previous item, $|S \cap G_1^M| = 2$. Let $S \setminus G_1^M = \{x_2\}$. Since $x_2 \in \mathrm{Nb}(G_1^M) \setminus G_1^M$, then there exists such an edge $x_1x_2 \in M$ that $x_1 \in T_1^M$. Then, clearly, S does not split G_2^M, thus, all vertices of the set $V(M) \setminus S$, except x_1, are in the same connected component of the graph $G - S$. Hence S splits T_1^M into exactly 2 parts, i.e. $|\mathrm{Part}(S)| = 2$. $\square$

Remark 13. 1) Let us consider in details the second case of the proof of previous lemma (when $S \setminus G_1^M = \{x_2\}$ and the cutset S separates x_1 from other vertices of the set $V(M)$). Obviously, the cutset S can be complement-

ed by an edge x_1x_2. Further two cases are possible: the set $S \cap T_1^M$ can be either empty or nonempty.

In the first case ($S \cap T_1^M = \varnothing$), clearly, x_1x_2 is a singular edge.

In the second case let $S \cap T_1^M = \{p\}$. Then the cutsets S, T_1^M and all cutsets of $\mathfrak{R}(M)$ containing p generate a flower. If $p \in M$, this flower contains M, else p is an end of an edge $e \in M$ and our flower contains all vertices of $V(M)$, except the other end of the edge e.

2) If the cut M is trivial ($G_2^M = \{x\}$) and S splits $V(M)$, then $x \in S$ and either S separates one vertex of the set T_1^M from two other vertices lying in the same connected component of the graph $G - S$, or the cut M is contained in a triple cut with line S.

Next lemma is about the neighborhood of a triple cut. Remind, that by definition the neighborhood of a triple cut $F = M_1 \cup M_2 \cup M_3$ is the set $\mathrm{Nb}(F) = V(M_1') \cup V(M_2') \cup V(M_3')$, where M_i' is a cut from $\mathfrak{M}_3(G)$ containing M_i if such cut exists, and M_i' coincides with M_i, otherwise.

Lemma 17. *Let a triple cut $F = M_1 \cup M_2 \cup M_3$ with a line S and a cutset $T \in \mathfrak{R}_3(G)$ be such, that $T \not\subset \mathrm{Nb}(F)$ and T splits $\mathrm{Nb}(F)$. Then $|\mathrm{Part}(T)| = 2$ and the cutset T is contained in some part $A_i \in \mathrm{Part}(S)$. Moreover, the cutset T separates a vertex $x_i \in \mathrm{Int}(A_i)$ from other vertices of the set $\mathrm{Nb}(F)$ and $T \setminus B_i' = \{x\}$ where $x \in S$, $xx_i \in M_i'$ and $B_i' \in \mathrm{Part}(M_i')$ is a part contained in A_i.*

Proof. Let T is dependent with S. Then by lemma 9 the set T is subordinated to S, thus, $T \subset V(F)$. We have a contradiction.

Hence, T is independent with S and, consequently, T is contained in a part of $\mathrm{Part}(S)$ (let it be A_i). Then, since T splits $\mathrm{Nb}(F)$ and $T \neq S$, the set T splits $V(M_i')$. Moreover, $T \not\subset V(M_i')$. If the cut M_i' is maximal, we apply lemma 16 to M_i' and to the cutset T. Since the cutsets S and T are independent, the statement 1° of lemma 16 cannot hold. Hence, the statement 2° of lemma 16 holds, that implies what is to be proved. If the cut M_i' is not maximal, then, by the definition, $M_i = M_i' \in \mathfrak{M}_1(G)$ and, since T is independent with S, the statement of our lemma in this case is clear. $\square$

Lemma 18. *Let a maximal flower $F = (p; q_1, \dots, q_m)$ and a cutset $T \in \mathfrak{R}_3(G) \setminus \mathfrak{R}(F)$ be such, that T splits $V(F)$. Then $|\mathrm{Part}(T)| = 2$ and one of two following statements hold.*

1° *The set T separates one vertex of the set $V(F)$ from other vertices of this set.*

2° *The set T separates two neighboring petals q_{i+1} and q_{i+2} from other vertices of the set $V(F)$. Moreover, $\mathrm{Int}(G_{i,i+1}) = \mathrm{Int}(G_{i+2,i+3}) = \varnothing$ and*

$T = \{q_i, x, q_{i+3}\}$, where $x \in \mathrm{Int}(G_{i+1,i+2})$ is the only vertex of the part $G_{i+1,i+2}$ adjacent to p.

Proof. Note, that by lemma 4 we have $p \notin T$. If the cutset T does not split $L = \{q_1, \ldots, q_m\}$, then T separates the center p from all petals of the flower, thus, statement 1° holds.

Let T split L. We shall prove that $T \cap L \neq \varnothing$. Indeed, if $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$ and $|T \cap \mathrm{Int}(G_{i,i+1})| \leq 1$ then by lemma 4 the petals q_i and q_{i+1} are connected in $G - T$. If $\mathrm{Int}(G_{i,i+1}) = \varnothing$, then the petals q_i and q_{i+1} are adjacent. Since there is not more than one such part $G_{j,j+1}$ that $|T \cap \mathrm{Int}(G_{j,j+1})| \geq 2$, all pairs of neighboring petals (except, maybe, one pair) are not splitted by T. Hence, if $T \cap L = \varnothing$, then all vertices of the set L are connected in $G - T$, we obtain a contradiction.

Note, that $|T \cap L| \leq 2$ by lemma 10. Consider the following two cases.

1. $T \cap L = \{q_i\}$.

In this case two other vertices of the cutset T must be in the same part of $\mathrm{Part}(F)$, otherwise, similarly to proved above, the cutset T does not split L. Let this part be $G_{j,j+1}$. Then it is clear, that the cutset T splits L into two sets $\{q_{i+1}, \ldots, q_j\}$ and $\{q_{j+1}, \ldots, q_{i-1}\}$. Without loss of generality we may assume, that p and $\{q_{j+1}, \ldots, q_{i-1}\}$ lie in the same connected component of the graph $G - T$. Let us prove, that $i + 1 = j$. Indeed, otherwise $T \cap \mathrm{Int}(G_{i,i+2}) = \varnothing$, consequently, the cutset T does not split the part $G_{i,i+2}$, i.e. p and q_{i+1} are connected in $G - T$, that contradicts our assumption. Hence, T separates the petal $q_{i+1} = q_j$ from other vertices of $V(F)$ and statement 1° holds.

2. $T \cap L = \{q_i, q_j\}$.

In this case, obviously, the cutset T splits L into sets $\{q_{i+1}, \ldots, q_{j-1}\}$ and $\{q_{j+1}, \ldots, q_{i-1}\}$. Without loss of generality we may assume, that p and $\{q_{j+1}, \ldots, q_{i-1}\}$ lie in the same connected component of the graph $G - T$. Let us prove, that the other set $\{q_{i+1}, \ldots, q_{j-1}\}$ consists of not more than two vertices. Indeed, otherwise $\mathrm{Int}(G_{i,i+2}) \cap \mathrm{Int}(G_{j-2,j}) = \varnothing$ and the cutset T cannot intersect interiors of both these parts. Then, similarly to proved above, p and $\{q_{i+1}, \ldots, q_{j-1}\}$ are connected in $G - T$, that contradicts our assumption.

Further, if $j = i + 3$, i.e. T separates the petals q_{i+1}, q_{i+2} from other vertices of $V(F)$, then among the parts $G_{i,i+1}$, $G_{i+1,i+2}$, $G_{i+2,i+3}$ is not more than one nonempty part (because each nonempty part contains a path, connecting its petal with the center of the flower and consisting of inner vertices of this part). Moreover, among these three parts there are no two neighboring empty parts, because by remark 2 their common petal is adjacent to the center. It is possible in the only case

$\mathrm{Int}(G_{i,i+1}) = \mathrm{Int}(G_{i+2,i+3}) = \varnothing$ and $\mathrm{Int}(G_{i+1,i+2}) \neq \varnothing$. Then, clearly, $T = \{q_i, x, q_{i+3}\}$ where $x \in \mathrm{Int}(G_{i+1,i+2})$. In addition, the cutsets T and $Q_{i+1,i+2}$ are dependent and $\{p, x\} \in \mathrm{Part}(\{T, Q_{i+1,i+2}\})$, hence, x is the only vertex of the part $G_{i+1,i+2}$ adjacent to p. Thus, statement 2° holds.

In all cases it is clear, that $|\mathrm{Part}(T)| = 2$, since the cutset T is dependent with some cutsets of F and splits each of them into exactly two parts. $\square$

Remark 14. In case 2° of previous lemma it is clear, that the cutset T is contained in a cut $M^*_{i,i+1} = \{q_iq_{i+1}, px, q_{i+3}q_{i+2}\}$.

Next lemmas will consider the case 1° of lemma 18 in details.

Lemma 19. *Let a maximal flower $F = (p; q_1, \dots, q_m)$ and a cutset $T \in \mathfrak{R}_3(G) \setminus \mathfrak{R}(F)$ be such that T separates a petal q_i from other vertices of the set $V(F)$. Then exactly one of two parts from* $\mathrm{Part}(F)$ *containing q_i is empty and the cutset T consists of the second petal of this part and two vertices of the other part containing q_i.*

Proof. Let $q_i \in H \in \mathrm{Part}(T)$. By the condition, q_i is the only vertex of $V(F)$ in $\mathrm{Int}(H)$. Then $\mathrm{Int}(H) \cap Q_{i-1,i+1} = \varnothing$, thus, $Q_{i-1,i+1}$ does not split H. That means $H \subset G_{i-1,i+1}$ and, in particular, $T \subset G_{i-1,i+1}$.

Note also, that $p \notin T$ by lemma 4 and, obviously, $q_i \notin T$. Thus one of the parts $G_{i-1,i}$ and $G_{i,i+1}$ contains not more than one vertex of the cutset T. Without loss of generality we may assume, that it is the part $G_{i-1,i}$. Clearly, $|T \cap G_{i-1,i}| = 1$, otherwise the vertices q_i and q_{i-1} are connected in $G - T$. Moreover, if $T \cap G_{i-1,i} \neq \{q_i\}$, then by lemma 4 there is a path connecting q_i and q_{i-1} in the part $G_{i-1,i}$ which do not intersect T. This is impossible. Thus, $T \cap G_{i-1,i} = \{q_{i-1}\}$, i.e. $T \cap \mathrm{Int}(G_{i-1,i}) = \varnothing$. But T separates from each other the vertices $p, q_i \in G_{i-1,i}$. It is possible only if $\mathrm{Int}(G_{i-1,i}) = \varnothing$. Then $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$, since otherwise by remark 2 the vertices q_i and p are adjacent. $\square$

Lemma 20. *Let a cutset $T \in \mathfrak{R}_3(G)$ separate a center of nondegenerate flower $F = (p; q_1, \dots, q_m)$ from other vertices of the set $V(F)$. Then T separates p from other vertices of the graph G. Moreover, $m \leq 6$, there are not more than three nonempty parts in* $\mathrm{Part}(F)$*, the interior of every nonempty part contains exactly one vertex of the set T, and the boundary of every nonempty part does not intersect T.*

Proof. Let $\mathrm{Part}(F)$ contain k nonempty parts and ℓ empty parts. If $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$, then $Q_{i,i+1}$ is a cutset dependent with T. Consequently, $T \cap \mathrm{Int}(G_{i,i+1}) \neq \varnothing$. Thus, $k \leq 3$. Further, if $\mathrm{Int}(G_{j-1,j}) = \mathrm{Int}(G_{j,j+1}) = \varnothing$, then by remark 2 the vertices p and q_j are adjacent, hence, $q_j \in T$. Note,

that empty parts of $\mathrm{Part}(F)$ are divided into not more than k sequences, which give us at least $\ell - k$ petals adjacent with p. Thus, $\ell = k + (\ell - k) \leq 3$, hence, $m = k + \ell \leq 6$.

Let $|T \cap G_{i,i+1}| = 2$. Then $|T \cap \mathrm{Int}(G_{i+1,i})| = 1$ and by lemma 5 the cutset $Q_{i,i+1}$ can be complemented by an edge px, where $T \cap \mathrm{Int}(G_{i+1,i}) = \{x\}$. This contradicts lemma 15.

Thus, $|T \cap G_{i,i+1}| \leq 1$ for every part $G_{i,i+1}$. Hence, if $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$, then $|T \cap \mathrm{Int}(G_{i,i+1})| = 1$ and $T \cap Q_{i,i+1} = \varnothing$. In addition, if $T \cap \mathrm{Int}(G_{i,i+1}) = \{u\}$, then $\{p, u\} \in \mathrm{Part}(\{T, Q_{i,i+1}\})$ by corollary 2, i.e. the cutset T separates p from other vertices of the part $G_{i,i+1}$. Since this condition holds for every nonempty part, then T separates p from other vertices of the graph G. $\square$

3.4 Singular flowers. The neighborhood of a flower

Definition 19. We call a flower $F = (p; q_1, \dots, q_m)$ *singular*, if $d(p) = 3$ and *nonsingular* otherwise. Let *neighborhood* of the center of this singular flower be the set $T(p)$ consisting of all adjacent to p vertices.

Note, that if p is the center of a singular flower, then $T(p) \in \mathfrak{R}_3(G)$. Moreover, by lemma 20 the interior of each nonempty part of $\mathrm{Part}(F)$ contains exactly one vertex of $T(p)$, and its boundary does not intersect $T(p)$.

Definition 20. Let $F = (p; q_1, \dots, q_m)$ be a maximal nondegenerate flower and $G_{i,i+1} \in \mathrm{Part}(F)$.

If the flower F is singular and the part $G_{i,i+1}$ is nonempty, then denote by $u_{i,i+1}$ the only vertex of $G_{i,i+1} \cap T(p)$.

If the flower F is nonsingular, there is exactly one vertex adjacent to p in the part $G_{i,i+1}$ and $\mathrm{Int}(G_{i-1,i}) = \mathrm{Int}(G_{i+1,i+2}) = \varnothing$. Then also denote by $u_{i,i+1}$ the only adjacent to p vertex of $G_{i,i+1}$.

In all other cases we set $u_{i,i+1} = p$.

The set $\mathrm{Nb}(F) = V(F) \cup \{u_{1,2}, u_{2,3}, \dots, u_{m,1}\}$ we call the *neighborhood* of the flower F.

Remark 15. Note, that if $u_{i,i+1} \neq p$, then the set $M_{i,i+1}$ can be complemented by an edge $pu_{i,i+1}$, i.e. $pu_{i,i+1} \in M^*_{i,i+1}$.

If F is a maximal nondegenerate singular flower, then by definition $\mathrm{Nb}(F) = V(F) \cup T(p)$.

If F is a maximal nondegenerate nonsingular flower and $u_{i,i+1} \neq p$, then by definition $\mathrm{Int}(G_{i-1,i}) = \mathrm{Int}(G_{i+1,i+2}) = \varnothing$, consequently, $M^*_{i,i+1} = \{q_{i-1}q_i, pu_{i,i+1}, q_{i+2}q_{i+1}\} \in \mathfrak{M}_3(G)$. On the other side, if $M^*_{i,i+1} \in \mathfrak{M}_3(G)$, then, clearly, $u_{i,i+1} \neq p$.

Definition 21. Let $u_{i,i+1} \neq p$. Set the notations $M'_{i,i+1} = M^*_{i,i+1}$ and $Q'_{i,i+1} = \{q_i, u_{i,i+1}, q_{i+1}\}$. If $u_{i,i+1} = p$, we set $M'_{i,i+1} = M_{i,i+1}$ and $Q'_{i,i+1} = Q_{i,i+1}$.

Let $G'_{i,i+1} = G_{i,i+1} \setminus \{p\}$ if either $u_{i,i+1} \neq p$, or $\mathrm{Int}(G_{i,i+1}) = \varnothing$ and at least one of the vertices $u_{i-1,i}$ and $u_{i+1,i+2}$ differs from p. In all other cases we set $G'_{i,i+1} = G_{i,i+1}$.

If $M'_{i,i+1} \in \mathfrak{M}(G)$, then denote by $\mathrm{Nb}(G'_{i,i+1})$ the neighborhood of $G'_{i,i+1}$ as of a part of $\mathrm{Part}(M'_{i,i+1})$. We call the cut $M'_{i,i+1}$ *boundary* cut of the part $G'_{i,i+1}$. If $M'_{i,i+1} \in \mathfrak{M}_0(G)$ we set $\mathrm{Nb}(G'_{i,i+1}) = G'_{i,i+1}$.

For $u_{i,i+1} \neq p$ it is easy to see that $G'_{i,i+1}$ is a part of $\mathrm{Part}(M'_{i,i+1})$ contained in $G_{i,i+1}$ and $\mathrm{Int}(G'_{i,i+1}) = \mathrm{Int}(G_{i,i+1}) \setminus \{u_{i,i+1}\}$.

Lemma 21. *Let $F = (p; q_1, \dots, q_m)$ be a maximal nondegenerate flower. Then the following statements hold.*

1) *If $T \in \mathfrak{R}_3(G)$ and $T \subset \mathrm{Nb}(F)$, then either $T \in \mathfrak{R}(F)$, or T is contained in $M'_{i,i+1}$ for some i, or $T = T(p)$ (the last is possible only for a singular flower F). All sets described above, except sets $Q'_{i,i+1}$, are cutsets splitting the graph G into exactly two parts. Each of these cutsets splits $\mathrm{Nb}(F)$. The set $Q'_{i,i+1}$ does not split $\mathrm{Nb}(F)$ and is a cutset if and only if $\mathrm{Int}(G'_{i,i+1}) \neq \varnothing$.*

2) *If a cutset $S \in \mathfrak{R}_3(G)$ splits $\mathrm{Nb}(F)$ and $S \not\subset \mathrm{Nb}(F)$, then $|\mathrm{Part}(S)| = 2$ and S separates one vertex of $\mathrm{Nb}(F)$ from the other vertices of this set. The vertex separated by S is not the center of F.*

Proof. 1) If $T \subset V(F)$, then by corollary 9 we have, that T is a set of the flower F. Then either $T \in \mathfrak{R}(F)$, or T is a boundary of a nonempty part $G_{i,i+1} \in \mathrm{Part}(F)$ and is contained in $M'_{i,i+1}$.

Let $T \not\subset V(F)$. Then by definition of the neighborhood of a flower there exists such i, that $u_{i,i+1} \in T$ and $u_{i,i+1} \neq p$. Note, that if T does not split $V(F)$, then $T \subset G_{i,i+1} \cap \mathrm{Nb}(F) = \{p, q_i, q_{i+1}, u_{i,i+1}\} \subset V(M'_{i,i+1})$. By corollary 10 we obtain, that T is contained in $M'_{i,i+1}$.

The only remaining case is $T \not\subset V(F)$ and T splits $V(F)$. By lemma 18 there are 3 possible subcases.

1. *T separates p from other vertices of $V(F)$.* Then by lemma 20 the flower F is singular and $T = T(p)$.

2. *T separates one petal of the flower F from other vertices of $V(F)$.* Since $T \cap I(G_{i,i+1}) \neq \varnothing$, then this petal is q_i or q_{i+1}. Then by lemma 19 two vertices of T belong to $G_{i,i+1} \cap \mathrm{Nb}(F) \subset V(M'_{i,i+1})$, and the third vertex belongs to a neighboring with $G_{i,i+1}$ empty part of $\mathrm{Part}(F)$, i.e. also belongs to $V(M'_{i,i+1})$. Thus, $T \subset V(M'_{i,i+1})$, hence, T is contained in $M'_{i,i+1}$.

3. *T separates petals q_i and q_{i+1} from other vertices of $V(F)$.* Thus, by lemma 18 we have $\mathrm{Int}(G_{i-1,i}) = \mathrm{Int}(G_{i+1,i+2}) = \varnothing$ and $T = \{q_{i-1}, u_{i,i+1}, q_{i+2}\} \subset M'_{i,i+1}$.

By the properties of a flower, every its inner set is a cutset, which splits the graph G into two parts and splits $V(F)$ (consequently, it also splits $\mathrm{Nb}(F)$). By lemma 6 the same statement holds for every inner set of the cut $M'_{i,i+1}$, and by lemma 20 — for the set $T(P)$ (if the flower F is singular). If $Q_{i,i+1} \neq Q'_{i,i+1}$, then, obviously, $Q_{i,i+1}$ is a cutset separating $u_{i,i+1}$ from other vertices of the set $\mathrm{Nb}(F)$ and, consequently, splits the graph G into exactly two parts. All remaining sets are of type $Q'_{i,i+1}$. Clearly, $Q'_{i,i+1}$ does not split $\mathrm{Nb}(F)$ and is a cutset if and only if $\mathrm{Int}(G'_{i,i+1}) \neq \varnothing$.

2) If S splits $V(F)$, then by lemma 4 we have $p \notin S$. Further, by lemma 18, the cutset S splits the graph into exactly two parts and separates not more than two vertices of the set $V(F)$ from other vertices of this set. If S separates two vertices, then $S \subset \mathrm{Nb}(F)$, that contradicts the condition. In addition, by lemma 20 the only 3-cutset separating p from other vertices of the set $V(F)$ is $T(p)$. But $T(p) \subset \mathrm{Nb}(F)$.

Hence, S separates one petal of the flower from other vertices of the set $V(F)$. In addition, since $p \notin S$, then all vertices $u_{i,i+1}$ and p belong to the same connected component of the graph $G - S$. Thus, S separates one petal of the flower from other vertices of the set $\mathrm{Nb}(F)$.

If S does not split $V(F)$, then S is independent with all sets of the flower F, i.e. S is contained in some part $G_{i,i+1}$. But then S can separate from the other vertices of the set $\mathrm{Nb}(F)$ not more than one vertex $u_{i,i+1}$, and it is possible only in the case $u_{i,i+1} \neq p$. Moreover, by lemma 2 the cutset S does not split $G_{i+1,i}$. In this case, clearly, $p \in S$, and every part of $\mathrm{Part}(S)$, which is a subset of $G_{i,i+1}$ contains a vertex adjacent to p. But, since $u_{i,i+1} \neq p$, there is only one such vertex in $G_{i,i+1}$, consequently, only one part of $\mathrm{Part}(S)$ is contained in $G_{i,i+1}$. On the other side, by lemma 2, the cutset S does not split $G_{i+1,i}$, hence, there is only one part of $\mathrm{Part}(S)$ not contained in $G_{i,i+1}$ i.e. $|\mathrm{Part}(S)| = 2$. □

Remark 16. Let us describe nondegenerate singular flower $F(p; q_1, \ldots, q_m)$ in details. Consider several cases.

1) Let $\mathrm{Part}(F)$ contain three nonempty parts. Clearly, $T(p)$ contains a vertex from the interior of each nonempty part, hence, p is not adjacent to any petal of F. Whence by remark 2 it follows, that no two empty parts of $\mathrm{Part}(F)$ are neighboring. Thus, $\mathrm{Part}(F)$ contains 4, 5 or 6 parts. A singular flower with 6 parts and three nonempty parts among them is shown on figure 6.

2) Let $\mathrm{Part}(F)$ contain two nonempty parts, then the interior of each nonempty part contains exactly one vertex of the set $T(p)$, and the third vertex of $T(p)$ is a petal q_i, not belonging to any nonempty part of $\mathrm{Part}(F)$ (i.e., the parts $G_{i-1,i}$ and $G_{i,i+1}$ are empty). Since all vertices adjacent to p be-

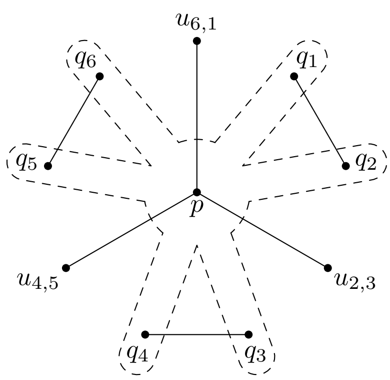

Figure 6: A singular flower with six parts and three nonempty parts

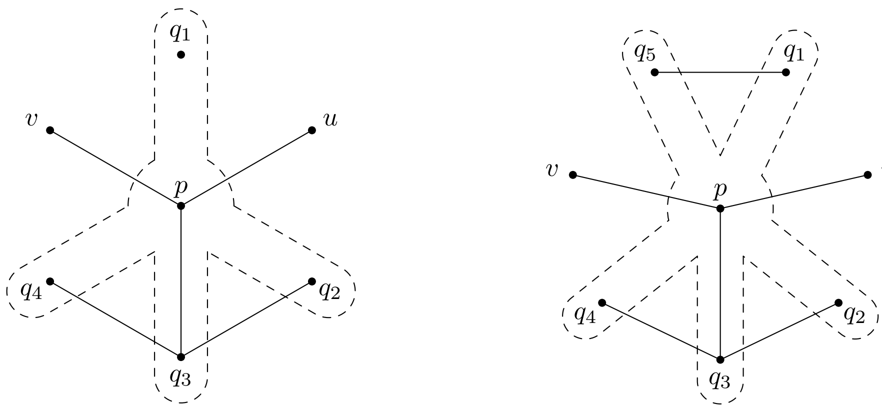

Figure 7: Singular petals with 4 or 5 parts and 2 nonempty parts

long to $T(p)$, the decomposition $\mathrm{Part}(F)$ except four parts mentioned above can contain not more than one empty part. Moreover, this empty part must be neighboring with two nonempty parts. Thus $\mathrm{Part}(F)$ consists of four or five parts.

If $|\mathrm{Part}(F)| = 4$, then this parts can be enumerated such that $G_{1,2} \ni u$ and $G_{4,1} \ni v$ are nonempty parts, $G_{2,3}$ and $G_{3,4}$ are empty parts, and $T(p) = \{u, v, q_3\}$. Note, that $F' = (q_3; q_1, u, p, v)$ is a flower of the same type, its center q_3 is separated from other vertices of the graph G by the set $\{q_2, p, q_4\}$. It is easy to see, that $\mathrm{Nb}(F) = \mathrm{Nb}(F')$.

If $|\mathrm{Part}(F)| = 4$, then the parts can be enumerated such, that $G_{1,2} \ni u$ and $G_{4,5} \ni v$ are nonempty parts, $G_{2,3}$, $G_{3,4}$ and $G_{5,1}$ are empty parts, and $T(p) = \{u, v, q_3\}$.

Note, that in this case $F' = (q_3; q_1, u, p, v, q_5)$ is a flower of the same type, its center q_3 is separated from other vertices of the graph G by the set $\{q_2, p, q_4\}$. It is easy to see, that $\mathrm{Nb}(F) = \mathrm{Nb}(F')$.

Singular flowers with 4 or 5 parts and 2 nonempty parts are shown on the figure 7.

3) The case when $\mathrm{Part}(F)$ contains exactly one nonempty part is impossible. Indeed, if it happens, the interior of this nonempty part contains a vertex $u \in T(p)$, and two other vertices of $T(p)$ are petals of our flower. There are at least three other parts of $\mathrm{Part}(F)$, all of them are empty. By remark 2 the common petal of two empty parts is adjacent to the center p, thus, this petal belongs to $T(p)$. Hence, there are exactly three empty parts in $\mathrm{Part}(F)$ — let them be $G_{1,2}$, $G_{2,3}$ and $G_{3,4}$. Now it is easy to see, that $\{up, q_1q_2, q_4q_3\} \in \mathfrak{M}_3(G)$ is a cut containing F. That contradicts maximality of F.

Lemma 22. *Let F be a singular nondegenerate flower. Then the following statements hold.*

1) *For every part $G_{i,i+1} \in \mathrm{Part}(F)$ we have $G'_{i,i+1} \neq G_{i,i+1}$.*

2) *The set $T(p)$ consists of all vertices $u_{i,i+1}$ different from p, and all vertices q_j for which $\mathrm{Int}(G_{j-1,j}) = \mathrm{Int}(G_{j,j+1}) = \varnothing$, $u_{j-2,j-1} \neq p$ and $u_{j+1,j+2} \neq p$.*

Proof. 1) Note, that in a singular flower $u_{i,i+1} \neq p$ (i.e. $G'_{i,i+1} \neq G_{i,i+1}$) if and only if $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$. Moreover, it follows from the classification of singular flowers (see remark 16), that for each empty part of $\mathrm{Part}(F)$ there is a nonempty neighboring part of $\mathrm{Part}(F)$. Thus, for empty parts of $\mathrm{Part}(F)$ we also have $G'_{i,i+1} \neq G_{i,i+1}$.

2) Note, that $T(p)$ consists of different from p vertices $u_{i,i+1}$ and all petals q_i belonging to two empty parts. It follows from remark 16, that there are no three empty consecutive parts in $\mathrm{Part}(F)$, thus, $q_i \in T(p)$ means, that $I(G_{i-1,i}) = I(G_{i,i+1}) = \varnothing$, $I(G_{i-2,i-1}) \neq \varnothing$ and $I(G_{i+1,i+2}) \neq \varnothing$. Then by item 1 we have $u_{i-2,i-1} \neq p$ and $u_{i+1,i+2} \neq p$. $\square$

3.5 A connection between triple cuts and other basic structures

A line of triple cut splits a graph into three parts and, consequently, it cannot be an inner set of a flower or a cut. Thus, the vertex set of a triple cut cannot be a subset of a vertex set of a flower or of a cut.

Moreover, it follows from lemmas 16 and 21, that a line of a triple cut cannot split a vertex set of a nontrivial cut or a neighborhood of a flower. Thus, if the vertex set of a nontrivial cut or of a flower is contained in the vertex set of a triple cut $F = M_1 \cup M_2 \cup M_3$ then it is contained in a vertex set of one of three cuts M_1, M_2, M_3. If a vertex set of a nontrivial cut or of a

flower is contained in $\mathrm{Nb}(F)$, then it is contained in the vertex set of one of three cuts M'_1, M'_2, M'_3. Note also, that edges connecting a vertex of degree 3 belonging to the line of a triple cut with three vertices of its neighborhood form a trivial cut which is contained in our triple cut.

Definition 22. We say, that a cut or a flower *is contained* in a triple cut, if its vertex set is contained in a vertex set of this triple cut.

We say, that a cut or a flower *is contained* in a neighborhood of a triple cut, if its vertex set is contained in this neighborhood.

4 Complexes

We represent the set $\mathfrak{R}_3(G)$ as a union of several subsets — structural units of decomposition, which are constructed on base of structures described above. We call these subsets *complexes*.

In this section we present all types of complexes, describe all 3-cutsets of each complex and the decomposition of the graph G by cutsets of one complex. Further with the help of theorem of decomposition [9] we construct a hypertree of relative position of different complexes. As a result we obtain a full description of relative position of all 3-cutsets in a triconnected graph.

Definition 23. We call a cutset $S \in \mathfrak{R}_3(G)$ *single*, if it is independent with any other cutset of $\mathfrak{R}_3(G)$. Otherwise, we call cutset S *nonsingle*.

A *single complex* is a complex consisting of one single cutset. Further we describe other types of complexes.

4.1 Triple complexes

Definition 24. For any triple cut F let the set consisting of all 3-cutsets contained in $\mathrm{Nb}(F)$, except boundaries of the neighborhood of F be a *triple complex*. Let the line of F be the *line* of this triple complex, and boundaries of the neighborhood of F be *boundaries* of this triple complex.

Let $F = M_1 \cup M_2 \cup M_3$ be a triple cut with line S, and $\mathrm{Nb}(F) = V(M'_1) \cup V(M'_2) \cup V(M'_3)$ be its neighborhood. Let $\mathrm{Part}(S) = \{A_1, A_2, A_3\}$, and parts $B_i \in \mathrm{Part}(M_i)$ and $B'_i \in \mathrm{Part}(M'_i)$ are such that $B'_i \subset B_i \subset A_i$. (All these parts are discussed in details in the Section 2.3.)

By lemma 12, the triple complex $\mathfrak{C}(F)$ consists of S, all trivial cutsets subordinated to S (there are not more than three such cutsets) and inner cutsets of the cuts M'_1, M'_2, M'_3.

Let us describe all parts of $\mathrm{Part}(\mathfrak{C}(F))$. If all cuts M'_i are nontrivial, then $\mathrm{Part}(\mathfrak{C}(F))$ consists of small parts $\{x, x_i\}$ (where $x \in S$ and $xx_i \in M'_i$) and normal parts B'_i.

If the cut M'_i is trivial, then $|B'_i| = 1$. Let $B'_i = \{y\}$. Then the part B_i consists of the vertex y and those vertices of the set S which degree is more, than three. If there are no such vertices, then all parts of $\mathrm{Part}(\mathfrak{C}(F))$, contained in A_i are parts $\{x, y\}$, where $x \in S$.

Let S contains a vertex of degree more, than 3. Then $B_i \in \mathrm{Part}(\mathfrak{C}(F))$. Moreover, the part B_i is small, if exactly two vertices of S has degree 3. Otherwise, there is exactly one vertex of degree three in S, in this case B_i is a normal part.

It follows from lemma 17, that for any set $R \in \mathfrak{R}_3(G) \setminus \mathfrak{C}(F)$ there exists a unique nonempty part $A \in \mathrm{Part}(\mathfrak{C}(F))$ such that $R \subset \mathrm{Nb}(A)$ and either $R = \mathrm{Bound}(A)$, or $R \cap \mathrm{Int}(A) \neq \varnothing$.

4.2 A complex of nondegenerate flower

Let there exists such a flower F in the graph G that all parts of $\mathrm{Part}(F)$ are empty. Then all vertices of the graph G are vertices of F, each petal is adjacent to the center and two neighboring petals. All 3-cutsets of the graph G are sets of the flower F. Note, that in this case the graph G is a "wheel" (see [10]). Further we assume, that for every flower F in the graph G the decomposition $\mathrm{Part}(F)$ contains a nonempty part.

In this section we consider a maximal nondegenerate flower $F = (p; q_1, q_2, \dots, q_m)$. As it was shown above, there are two essentially different cases: the flower F can be singular of nonsingular.

Definition 25. Let a *complex* $\mathfrak{C}(F)$ of a flower F be the set of all 3-cutsets contained in its neighborhood $\mathrm{Nb}(F)$ which split $\mathrm{Nb}(F)$. We call *boundaries* of the complex $\mathfrak{C}(F)$ boundaries of all normal parts of $\mathrm{Part}(\mathfrak{C}(F))$.

It follows from lemma 21, that $\mathfrak{C}(F)$ consists of cutsets of $\mathfrak{R}(F)$, cutsets contained in cuts $M'_{i,i+1}$ and not coinciding with $Q'_{i,i+1}$ (here it is enough to consider only such i for which $u_{i,i+1} \neq p$) and, if the flower F is singular, the cutset $T(p)$. Boundaries of the complex of nondegenerate flower do not belong to this complex!

Lemma 23. *Let $G_{i,i+1} \in \mathrm{Part}(\mathfrak{R}(F))$ and $G'_{i,i+1} \neq G_{i,i+1}$. Then there exists a cutset $S_{i,i+1} \in \mathfrak{C}(F)$ separating p from $G'_{i,i+1}$. Moreover, $S_{i,i+1} \cap G_{i,i+1} = \{u_{i,i+1}\}$ if $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$ and $S_{i,i+1} \cap G_{i,i+1}$ consists of one of the petals q_i and q_{i+1}, otherwise.*

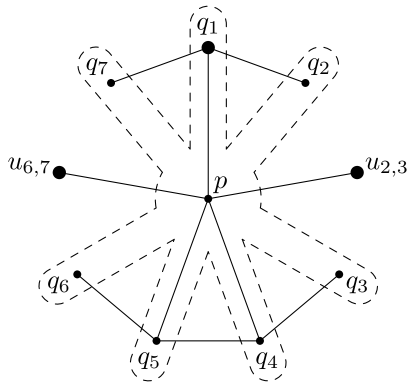

Figure 8: The neighborhood of the center of nonsingular flower

Proof. If $\mathrm{Int}(G_{i,i+1}) = \varnothing$, then, since $G'_{i,i+1} \neq G_{i,i+1}$, at least one of the vertices $u_{i-1,i}$ and $u_{i+1,i+2}$ does not coincide with p. Without loss of generality, let it be $u_{i-1,i}$. Then the set $S_{i,i+1} = \{q_{i-1}, u_{i-1,i}, q_{i+1}\}$ is what we want. If $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$, then $S_{i,i+1} = T(p)$ for a singular flower and $S_{i,i+1} = \{q_{i-1}, u_{i,i+1}, q_{i+2}\}$ for a nonsingular flower is the desired set. $\square$

To describe parts of decomposition of the graph by a complex of flower we need to generalize the notion of the neighborhood of the center for the case of nonsingular flower. For this purpose we get help of the property of the neighborhood of the center of a singular flower, proved in item 2 of lemma 22.

Definition 26. Let the *neighborhood* of the center of a flower F be a set $T(p)$ consisting of all vertices $u_{i,i+1}$ different from p, and all petals q_j for which $\mathrm{Int}(G_{j-1,j}) = \mathrm{Int}(G_{j,j+1}) = \varnothing$, $u_{j-2,j-1} \neq p$ and $u_{j+1,j+2} \neq p$.

For example, the neighborhood of the center of the flower shown on figure 8 consists of vertices q_1, $u_{2,3}$ and $u_{6,7}$. The petals q_4 and q_5 do not belong to the neighborhood, since $u_{5,6} = u_{3,4} = p$.

Lemma 24. *The set* $\mathrm{Part}(\mathfrak{C}(F))$ *consists of all parts* $G'_{i,i+1}$ *and small parts* $\{p, x\}$*, where* $x \in T(p)$*. A part* $G'_{i,i+1}$ *is small if and only if* $\mathrm{Int}(G_{i,i+1}) = \varnothing$ *and at least one of the vertices* $u_{i-1,i}$ *and* $u_{i+1,i+2}$ *is different from* p*. In addition, if the part* $G'_{i,i+1}$ *is normal, then* $\mathrm{Bound}(G'_{i,i+1}) = Q'_{i,i+1}$.

Proof. By the definition $\mathrm{Part}(\mathfrak{R}(F)) = \mathrm{Part}(F) = \{G_{1,2}, G_{2,3}, \ldots, G_{m,1}\}$. Let us see, how other sets of $\mathfrak{C}(F)$ can split the parts of $\mathrm{Part}(F)$.

At first we shall prove, that $G'_{i,i+1} \in \mathrm{Part}(\mathfrak{C}(F))$ for all i. Note, that if $G'_{i,i+1} \neq G_{i,i+1}$, then by lemma 23 there exists a cutset $S_{i,i+1} \in \mathfrak{C}(F)$ separating p from $G'_{i,i+1}$. Thus, it is enough to prove, that no set of $\mathfrak{C}(F)$ splits $G'_{i,i+1}$. Consider several cases.

1. Let $G'_{i,i+1} = G_{i,i+1}$. Then by lemma 22 the flower F is nonsingular.

1.1. Let $\mathrm{Int}(G_{i,i+1}) = \varnothing$. Then $u_{i-1,i} = u_{i+1,i+2} = p$ and q_i is adjacent to q_{i+1}. Hence, the cutset T, splitting $G'_{i,i+1}$, must separate p from $\{q_i, q_{i+1}\}$. Without loss of generality we may assume, that $q_i \notin T$. Thus, by lemmas 18 and 19 we have $T \cap \mathrm{Int}(G_{i-1,i}) \neq \varnothing$, whence $T \notin \mathfrak{C}(F)$, since $u_{i-1,i} = p$.

1.2. Let $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$. Since $u_{i,i+1} = p$, then no cutset of $\mathfrak{C}(F)$ intersect $\mathrm{Int}(G_{i,i+1})$ or coincide with $Q_{i,i+1}$. But $G_{i,i+1}$ is a union of several parts of $\mathrm{Part}(Q_{i,i+1})$. Hence, by lemma 2, no cutset of $\mathfrak{C}(F)$ can split $G_{i,i+1} = G'_{i,i+1}$.

2. Let $G'_{i,i+1} \neq G_{i,i+1}$. The case $\mathrm{Int}(G_{i,i+1}) = \varnothing$ is clear, since in this case the vertices q_i and q_{i+1} are adjacent and no cutset can split $G'_{i,i+1} = \{q_i, q_{i+1}\}$. Note, that it is the only case when the part $G'_{i,i+1}$ is small. Thus, it is enough to consider the case $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$. We divide it into two subcases.

2.1. If $\mathrm{Int}(G'_{i,i+1}) \neq \varnothing$, then the part $G'_{i,i+1}$ is a union of several parts of $\mathrm{Part}(Q'_{i,i+1})$. Since no cutset of $\mathfrak{C}(F)$ intersect $\mathrm{Int}(G'_{i,i+1})$ or coincide with $Q'_{i,i+1}$, then by lemma 2 no set of $\mathfrak{C}(F)$ can split $G'_{i,i+1}$.

2.2. If $\mathrm{Int}(G'_{i,i+1}) = \varnothing$, then $\mathrm{Int}(G_{i,i+1}) = \{u_{i,i+1}\}$, i.e. the vertex $u_{i,i+1}$ is adjacent to q_i and q_{i+1}. Further, by lemma 4 there exist a path between q_i and q_{i+1}, avoiding $u_{i,i+1}$, which inner vertices belong to $\mathrm{Int}(G_{i,i+1})$. Hence, the vertices q_i and q_{i+1} are also adjacent and no cutset can split $G'_{i,i+1} = \{q_i, u_{i,i+1}, q_{i+1}\}$.

Now all the sets $G'_{i,i+1}$ belong to $\mathrm{Part}(\mathfrak{C}(F))$. It follows from the definition of $G'_{i,i+1}$ and $Q'_{i,i+1}$, that $\mathrm{Bound}(G'_{i,i+1}) = Q'_{i,i+1}$ if the part $G'_{i,i+1}$ is normal.

Note, that all the sets $\{p, x\}$ where $x \in T(p)$ also belong to $\mathrm{Part}(\mathfrak{C}(F))$. Indeed, p and x are adjacent and $\{p, x\}$ can be separated from other vertices of the graph G by several cutsets of $\mathfrak{C}(F)$. If $x = u_{i,i+1}$, then the sets $Q_{i,i+1}$ and $S_{i,i+1}$ (which was constructed in lemma 23) fit for this purpose. Otherwise, if $x = q_j$, then we use the sets $Q_{j-1,j+1}$, $S_{i-1,i}$ and $S_{i,i+1}$.

Prove, that there are no other parts. Let $H \in \mathrm{Part}(\mathfrak{C}(F))$ be another part. Clearly, the part H is contained in one of the sets $G_{i,i+1}$ and $H \not\subset G'_{i,i+1}$, hence, $G'_{i,i+1} \neq G_{i,i+1}$ and $p \in H$.

If $\mathrm{Int}(G_{i,i+1}) \neq \varnothing$, then $u_{i,i+1} \neq p$ and $H = \{p, u_{i,i+1}\}$, since by lemma 23 there exists a set $S_{i,i+1} \in \mathfrak{C}(F)$ separating p from other vertices of $G_{i,i+1}$.

If $\mathrm{Int}(G_{i,i+1}) = \varnothing$, then either $H = \{p, q_i\}$, or $H = \{p, q_{i+1}\}$. Without loss of generality we assume, that $H = \{p, q_i\}$. Then $H \subset G_{i-1,i}$, hence, $\mathrm{Int}(G_{i-1,i}) = \varnothing$ (otherwise we get help of proved above), i.e. the vertices p and q_i are adjacent. Further, by lemma 23 there exist sets $S_{i-1,i}$ and $S_{i,i+1}$, separating p from q_{i-1} and q_{i+1} respectively. Then by lemmas 18 and 19 we have $S_{i-1,i} \cap \mathrm{Int}(G_{i-2,i-1}) \neq \varnothing$ and $S_{i,i+1} \cap \mathrm{Int}(G_{i+1,i+2}) \neq \varnothing$, hence, $u_{i-2,i-1} \neq p$ and $u_{i+1,i+2} \neq p$. That means $q_i \in T(p)$. $\square$

For every normal part of $\mathrm{Part}(\mathfrak{C}(F))$ we define its neighborhood.

Definition 27. For every normal part $G'_{i,i+1} \in \mathrm{Part}(F)$ let its neighborhood be the set $\mathrm{Nb}(G'_{i,i+1}) = G'_{i,i+1} \cup V(M'_{i,i+1})$.

Remark 17. If $M'_{i,i+1} = Q_{i,i+1}$, then $\mathrm{Nb}(G'_{i,i+1}) = G'_{i,i+1} = G_{i,i+1}$. In all other cases the normal part $G'_{i,i+1} \in \mathrm{Part}(\mathfrak{C}(F))$ has neighboring cut $M'_{i,i+1}$. Then the neighborhoods of $G'_{i,i+1}$ as a part of $\mathrm{Part}(\mathfrak{C}(F))$ and as a part of $\mathrm{Part}(M'_{i,i+1})$ coincide.

Theorem 3. *Let $F = (p; q_1, \dots, q_m)$ be a maximal nondegenerate flower, $R \in \mathfrak{R}_3(G) \setminus \mathfrak{C}(F)$. Then there exists a unique nonempty part $H \in \mathrm{Part}(\mathfrak{C}(F))$ such that $R \subset \mathrm{Nb}(H)$ and either $R = \mathrm{Bound}(H)$, or $R \cap \mathrm{Int}(H) \neq \varnothing$.*

Proof. If $R \subset \mathrm{Nb}(F)$, then by lemmas 21 and 24 the set R is a boundary of a nonempty part $H \in \mathrm{Part}(\mathfrak{C}(F))$. Hence $R \subset H \subset \mathrm{Nb}(H)$. Clearly, a boundary of a nonempty part of $\mathrm{Part}(\mathfrak{C}(F))$ neither is a boundary nor intersect the interior of another nonempty part of $\mathrm{Part}(\mathfrak{C}(F))$.

Let $R \not\subset \mathrm{Nb}(F)$. Then there exists a part $G'_{i,i+1} \in \mathrm{Part}(\mathfrak{C}(F))$ such that $R \cap \mathrm{Int}(G'_{i,i+1}) \neq \varnothing$. Prove, that $R \subset \mathrm{Nb}(G'_{i,i+1})$. It is obvious in the case when R is independent with $Q'_{i,i+1}$, since then $R \subset G'_{i,i+1}$. Hence it is enough to consider the case when these two sets are dependent. In this case by lemma 21 the set R separates one vertex $x \in \mathrm{Nb}(F)$ from other vertices of $\mathrm{Nb}(F)$. Obviously, $x \in Q'_{i,i+1} = \{q_i, u_{i,i+1}, q_{i+1}\}$. There are two possible cases.

1. If $x = q_i$ (the case $x = q_{i+1}$ is similar), then by lemma 19 the set R consists of two vertices of the part $G_{i,i+1} \subset \mathrm{Nb}(G'_{i,i+1})$ and a vertex $q_{i-1} \in \mathrm{Nb}(G'_{i,i+1})$.

2. Otherwise, $x = u_{i,i+1}$. In this case $u_{i,i+1} \neq p$. Since $p \in \mathrm{Nb}(F)$ and p is adjacent to $u_{i,i+1}$, then $p \in R$. Thus R is independent with $Q_{i,i+1}$ (otherwise by lemma 4 the flower F is not maximal). Hence, $R \subset G_{i,i+1} \subset \mathrm{Nb}(G'_{i,i+1})$.

It remains to notice, that the neighborhood of another part of $\mathrm{Part}(\mathfrak{C}(F))$ does not intersect $\mathrm{Int}(G_{i,i+1})$ and, consequently, does not contain R. $\square$

4.3 A complex of big cut

Definition 28. 1) We call a nontrivial cut $M \in \mathfrak{M}_3$ *big*, if $V(M)$ is not a subset of the neighborhood of any triple cut or nondegenerate flower.

2) Define the *complex* $\mathfrak{C}(M)$ of a big cut M as a set of all 3-cutsets contained in $V(M)$, except boundaries of this cut T_1^M and T_2^M, which we call *boundaries* of $\mathfrak{C}(M)$.

By corollary 10 every set $R \in \mathfrak{C}(M)$ is contained in the cut M (i.e., contains a vertex of each edge of the cut M). All such sets, except boundaries of M, belong to $\mathfrak{C}(M)$. As we know, $\mathrm{Part}(\mathfrak{C}(M))$ consists of normal parts G_1^M and G_2^M, and small parts $\{x_1, x_2\}$ where $x_1x_2 \in M$. We set, that the neighborhood of G_i^M as a part of $\mathrm{Part}(\mathfrak{C}(M))$ is its neighborhood as a part of $\mathrm{Part}(M)$.

It follows from lemma 16, that for any set $R \in \mathfrak{R}_3(G) \setminus \mathfrak{C}(M)$ there exists a unique nonempty part $A \in \mathrm{Part}(\mathfrak{C}(M))$ such that $R \subset \mathrm{Nb}(A)$ and either $R = \mathrm{Bound}(A)$, or $R \cap \mathrm{Int}(A) \neq \varnothing$.

Let $M = \{a_1a_2, b_1b_2, c_1c_2\}$ be a big cut. As it was proved above, there exist six four-petal flowers on vertices of $V(M)$ (they are $(b_1; a_1, a_2, c_2, c_1)$ and five similar flowers). Since the cut M is not contained in the neighborhood of a nondegenerate flower, all these flowers are maximal and, of course, degenerate.

4.4 Small complexes

Definition 29. Let triple complexes, complexes of nondegenerate flower and complexes of big cut be *big* complexes. All cutsets not belonging to any big complex we shall divide into complexes of one or two cutsets. We call such complexes *small*.

Let the vertex set of each (big or small) complex $\mathcal{C}$ be the union $V(\mathcal{C})$ of all cutsets of $\mathcal{C}$.

Each single cutset form a single complex (which is small). Further we describe other small complexes.

Let $T \in \mathfrak{R}_3(G)$ be a nonsingle cutset, not belonging to any big complex. Note, that then $|\mathrm{Part}(T)| = 2$. Indeed, otherwise by lemma 9 any 3-cutset dependent with T is subordinated to T, i.e. T is a line of triple complex. In addition, if cutset $S \in \mathfrak{R}_3(G)$ is dependent with T, then $|\mathrm{Part}(S)| = 2$ (otherwise T is subordinated to S and belongs to a triple complex with line S) and $T \cap S = \varnothing$ (otherwise the sets T and S generate a flower).

Lemma 25. *Let a cutset $T = \{x, y, z\}$ be such that $|\mathrm{Part}(T)| = 2$ and T can be complemented by each of edges xx_1 and yy_1, lying in different parts of* $\mathrm{Part}(T)$. *Then T can be complemented by both these edges simultaneously (i. e. $\{xx_1, y_1y, z\} \in \mathfrak{M}_2(G)$) if and only if the vertices x and y are not adjacent.*

Proof. Let the vertex x_1 lie in a connected component H of the graph $G - T$. By lemma 5 we know, that x_1 is the only vertex of the component H adjacent to x. Further consider a cut $M_y = \{x, y_1y, z\} \in \mathfrak{M}_1(G)$. Obviously, $H \cup \{y\}$

is a connected component of the graph $G - M_y$. By lemma 5 the cut M_y can be complemented by an edge xx_1 if and only if x_1 is the only vertex of the set $H \cup \{y\}$, adjacent to x. The last fact is equivalent to that x and y are not adjacent. □

Lemma 26. *Let $T = \{x, y, z\}$ be a nonsingle cutset not belonging to any big complex. Then the following statements hold.*

1) *For any cutset $S \in \mathfrak{R}_3(G)$ dependent with T exactly one part of* $\mathrm{Part}(\{S, T\})$ *is small. Vertices of this part form a singular edge, both cutsets S and T can be complemented by this edge.*

2) *All edges which complement the cutset T lie in the same part of* $\mathrm{Part}(T)$. *Moreover, the set T can be complemented by all these edges simultaneously.*

Proof. 1) As it was proved above, $T \cap S = \varnothing$, i.e. by corollary 2 at least one part of $\mathrm{Part}(\{S, T\})$ is small. From $\mathrm{Part}(T) = \mathrm{Part}(S) = 2$ it follows, that there is exactly one small part. By theorem 2 we know, that vertices of this part form a singular edge, by which both cutsets S and T can be complemented.

2) At first notice, that the cutset T cannot be complemented by two edges, lying in the different parts of $\mathrm{Part}(T)$ simultaneously. Indeed, otherwise T is an inner set of a cut of $\mathfrak{M}_2(G)$, i.e. belongs to a big complex.

Now suppose, that the cutset T can be complemented by each of edges xx_1 and yy_1, and these two edges lie in different parts of $\mathrm{Part}(T)$. Then by lemma 25 the vertices x and y are adjacent. Consider a set $S \in \mathfrak{R}_3(G)$ dependent with T (such a set exists, since T is nonsingle). We know, that $T \cap S = \varnothing$, hence, S separates z from $\{x, y\}$. Then from item 1 of this lemma it follows, that the set T can be complemented by a singular edge zz_1. Without loss of generality assume, that y_1 and z_1 are in different connected components of the graph $G - T$. Then, similarly to proved above, the vertices y and z are adjacent and the set S cannot split T, we obtain a contradiction.

If the set T can be complemented by edges xx_1 and xx_2, then by lemma 5 each of the vertices x_1 and x_2 is the only vertex adjacent to x in the connected component of the graph $G - T$ containing this vertex. Since $d(x) \geq 3$, it follows, that x is adjacent to a vertex $y \in T$ and similarly to written above we obtain a contradiction.

Thus, all edges, by which we can complement the set T are in the same part of $\mathrm{Part}(T)$. Then by lemma 5 it follows, that the cutset T can be complemented by all these edges simultaneously. □

Now we can describe all types of small complexes, parts of decomposition of the graph by a small complex and neighborhoods of these parts.

Definition 30. For any cut $M = \{x_1x_2, y, z\}$ which boundaries are nonsingle cutsets not belonging to any complex we define the complex $\mathfrak{C}(M)$ as a set consisting of both boundaries of M. We call the cut M *small*, and complex $\mathfrak{C}(M)$ — a *complex of small cut*.

All other cutsets, not belonging to big complexes or complexes of small cuts, form complexes consisting of one cutset.

It is easy to see, that the complex of small cut $M = \{x_1x_2, y, z\}$ splits G into three parts: G_1^M, G_2^M and $\{x_1, x_2, y, z\}$. All these parts are normal. But the part $\{x_1, x_2, y, z\}$ is splitted by cutsets dependent with boundaries of M, and there is a small part $\{x_1, x_2\} \in \mathrm{Part}(\mathfrak{R}_3(G))$. This part is also empty. We set, that neighborhoods of G_i^M as a part of $\mathrm{Part}(\mathfrak{C}(M))$ and as a part of $\mathrm{Part}(M)$ coincide.

For every small complex $\mathcal{C} = \{T\}$ let us define the neighborhood of a part of $\mathrm{Part}(\mathcal{C})$. If T is a single set, then for every part $H \in \mathrm{Part}(\mathcal{C})$ we set $\mathrm{Nb}(H) = H$. Otherwise, let $\mathrm{Part}(\mathcal{C}) = \{H_1, H_2\}$. By lemma 26, the ends of all edges which complement the set T are in the same part of $\mathrm{Part}(\mathcal{C})$ — let it be H_1. We set $\mathrm{Nb}(H_1) = H_1$. Let us define neighborhood of the other part H_2. We complement the cutset T to a maximal cut M. Clearly, $H_2 \in \mathrm{Part}(M)$. Let the neighborhood of H_2 as a part of $\mathrm{Part}(\mathcal{C})$ be its neighborhood as a part of $\mathrm{Part}(M)$.

It follows from lemma 16, that for any small complex $\mathcal{C}$ and any cutset $R \in \mathfrak{R}_3(G) \setminus \mathcal{C}$ there exists a unique nonempty part $A \in \mathrm{Part}(\mathcal{C})$ such that $R \subset \mathrm{Nb}(A)$ and $R \cap \mathrm{Int}(A) \neq \varnothing$.

Let us describe all small complexes in details.

Lemma 27. *Let $T = \{x, y, z\}$ be a nonsingle cutset not belonging to any big complex. Then at least one of the following three statements holds.*

1° *The cutset T is trivial.*

2° *The cutset T is a boundary of a big complex. All cutsets of this complex and all edges which complement the set T lie in the same part of* $\mathrm{Part}(T)$.

3° *Exactly one edge xx_1 complements the cutset T, this edge is singular, and each cutset dependent with T contains x_1 and separates x from $\{y, z\}$.*

Proof. Since the cutset T is nonsingle, then there exists a cutset $S \in \mathfrak{R}_3(G)$ dependent with T. By lemma 26 there is one small part in $\mathrm{Part}(\{S, T\})$ and its two vertices are ends of a singular edge, which complements the cutsets S and T. Without loss of generality we may assume, that this edge is xx_1.

Further we consider several cases.

1. *Let xx_1 be the only edge which complements T.* Then for every set $R \in \mathfrak{R}_3(G)$ dependent with T we have $\{x, x_1\} \in \mathrm{Part}(\{R, T\})$. Hence, $x_1 \in R$ and R separates x from $\{y, z\}$. Thus in this case statement 3° holds.

2. *Let an edge yx_1 also complement the cutset T.* Denote by H a connected component of the graph $G-T$ containing x_1. By lemma 26, the set T can be complemented by edges yx_1 and xx_1 simultaneously. Then by item 2 of remark 4 we obtain, that $H = \{x_1\}$ and T is a trivial cutset. In this case statement 1° holds.

3. *Let an edge yy_1 also complement the cutset T (where $x_1 \neq y_1$).* Then by lemma 26 the cutset T can be complemented by the edges xx_1 and yy_1 simultaneously, i.e. $M = \{xx_1, yy_1, z\} \in \mathfrak{M}_2(G)$. Inner sets of the cut M generate a flower $(z; x, x_1, y_1, y)$, which is contained in some big complex $\mathcal{C}$. Then $T \subset V(\mathcal{C})$. But by condition of lemma $T \notin \mathcal{C}$, hence, the cutset T is a boundary of the complex $\mathcal{C}$. Since a boundary of a complex cannot split its vertex set, then $V(\mathcal{C})$ is contained in the part of $\mathrm{Part}(T)$ which contains the vertices x_1 and y_1. In this case statement 2° holds. $\square$

Remark 18. 1) The statement 1° cannot be fulfilled simultaneously with one of statements 2° or 3°.

2) It is easy to see from the prove of lemma 27, that if a nonsingle 3-cutset not belonging to any big complex can be complemented by exactly one edge, then statement 3° of lemma 27 holds for this cutset.

3) A boundary of a triple complex or of a complex of big cut always can be complemented by an edge, lying in the part containing all vertices of this complex. A boundary $Q'_{i,i+1}$ of a complex of flower $\mathfrak{C}(F)$ cannot be complemented by such edge in the only case: the flower F is nonsingular, $u_{i,i+1} = p$ and both parts $G_{i-1,i}$ and $G_{i+1,i+2}$ are nonempty. It is easy to see, that in this case the set $Q'_{i,i+1}$ is single.

It follows from written above, that if a boundary of a big complex is not a single cutset and do not belong to another big complex, then it cannot be complemented by an edge lying in the part not containing all vertices of this complex.

In particular, all cutsets, belonging to a complex of small cut cannot be boundaries of big complexes. Clearly, such cutsets also cannot be trivial or single. Hence, each set of the complex of a small cut $M = \{x_1x_2, y, z\}$ can be complemented by an edge x_1x_2.

Lemma 28. *Let $T = \{x, y, z\}$ be a nonsingle cutset, which do not belong to any big complex and is not a boundary of big complex. Let T can be complemented by exactly one edge xx_1. Then $T_1 = \{x_1, y, z\}$ is a cutset and one of two following statements holds.*

1° *The cutset T_1 is single.*

2° *Two cutsets T and T_1 form a complex of small cut.*

Proof. The set T_1 can be not a cutset in the only case: if $\{x_1\}$ is a connected

component of the graph $G - T$. But then the cutset T is trivial and can be complemented by edges yx_1 and zx_1 too. We obtain a contradiction.

Thus, T_1 is a cutset. Suppose, that it is nonsingle. We need to prove, that in this case the cutsets T and T_1 form a complex of small cut, i.e. T_1 does not belong to any big complex.

Note, that by lemma 27 the edge xx_1 is singular and any 3-cutset S dependent with T contains x_1 and separates x from $\{y, z\}$. Hence, S cannot be dependent with T_1. Thus, every 3-cutset dependent with T_1 is independent with T.

Now assume, that T_1 belongs to a big complex $\mathcal{C}$. Then T_1 must be dependent with at least one cutset $S \in \mathcal{C}$. Since S is dependent with T_1 and independent with T, it must contain the vertex x. But then $T \subset V(\mathcal{C})$, i.e. either T belongs to $\mathcal{C}$, or T is a boundary of $\mathcal{C}$. We obtain a contradiction. $\square$

Corollary 12. *Let a complex consist of one cutset. Then this cutset can be:*

1) *a single cutset;*
2) *a trivial cutset;*
3) *a boundary of big complex;*
4) *a cutset which can be complemented by exactly one edge, and the other boundary of resulting cut is a single cutset.*

5 Relative position of complexes

In previous section we have described all types of complexes and for each type we have investigated some properties. Let us repeat properties, that hold for all types of complexes.

For every complex $\mathcal{C}$ a boundary of any nonempty part $A \in \mathrm{Part}(\mathcal{C})$ is a 3-cutset, which do not split $V(\mathcal{C})$ (but can split A). Let $R = \mathrm{Bound}(A)$. If $\mathcal{C}$ consists of more than one cutset, then $\mathrm{Part}(R)$ contains exactly one part which do not intersect $\mathrm{Int}(A)$. Denote this part by $\overline{A}$. In this case the neighborhood of A is constructed as follows: the cutset R is complemented by all possible edges lying in $\overline{A}$. Let M be the resulting cut. After that we set, that $\mathrm{Nb}(A)$ is a neighborhood of A as a part of $\mathrm{Part}(M)$. Note also, that if $A \in \mathrm{Part}(\mathcal{C}_1)$ and $A \in \mathrm{Part}(\mathcal{C}_2)$ (it is possible, for example, if $\mathcal{C}_1$ is a big complex and $\mathcal{C}_2 = \{R\}$, where R is a bound of $\mathcal{C}_1$ and $|\mathrm{Part}(R)| = 2$), then neighborhoods of A in both cases coincide.

Definition 31. For any complex $\mathcal{C}$ and any cutset $T \in \mathfrak{R}_3(G) \setminus \mathcal{C}$ we say, that T *belongs* to a nonempty part $A \in \mathrm{Part}(\mathcal{C})$, if $T \subset \mathrm{Nb}(A)$ and either $T = \mathrm{Bound}(A)$, or $T \cap \mathrm{Int}(A) \neq \varnothing$.

In previous section it was proved, that for any complex $\mathcal{C}$ any cutset $T \in \mathfrak{R}_3(G) \setminus \mathcal{C}$ belongs to exactly one nonempty part of $\mathrm{Part}(\mathcal{C})$.

Our first aim is to show, that two cutsets belonging to one complex cannot belong to different parts of decomposition of the graph by another complex. For this purpose we need the following lemmas.

Lemma 29. *Let $\mathcal{C}$ be a complex and $T \in \mathfrak{R}_3(G) \setminus \mathcal{C}$ be a set splitting $V(\mathcal{C})$. Let T belong to a part $A \in \mathrm{Part}(\mathcal{C})$ and $R = \mathrm{Bound}(A)$. Then the following statements hold.*

1) *The cutsets R and T are dependent.*

2) *The cutset T separates exactly one vertex $x \in R$ from other vertices of $V(\mathcal{C})$.*

3) *The cutset T consists of two vertices of the part A and a vertex $y \notin A$ such that both cutsets R and T can be complemented by the edge xy.*

Proof. 1) Note, that $T \not\subset A$, since otherwise T cannot split $V(\mathcal{C})$. Hence, $T \neq R$. Since T belongs to the part A we obtain, that $T \cap \mathrm{Int}(A) \neq \varnothing$. Thus, R splits T, consequently, these sets are dependent.

2) It follows from previous lemmas, that the cutset T separates exactly one vertex $x \in V(\mathcal{C})$ from other vertices of the set $V(\mathcal{C})$: for a complex of big or small cut it follows from lemma 16, for a triple complex — from lemma 17, for a complex of flower — from lemma 21. In the case $|\mathcal{C}| = 1$ this statement is obvious. Since T and R are dependent, then $x \in R$.

3) Add the cutset R to a cut M by all possible edges lying in the part $\overline{A}$ (remind, that $\overline{A}$ is the only part of $\mathrm{Part}(R)$ not intersecting $\mathrm{Int}(A)$). Let M' be a maximal cut containing M. Since $T \subset \mathrm{Nb}(A) = A \cup V(M)$, the cutset T cannot generate a flower with both boundaries of the cut M'. If $T \not\subset V(M')$, then statement 3 of our lemma follows from item 2 of lemma 16 for the cut M' and the cutset T. If $T \subset V(M')$, then we have $M' \neq M$. It is possible only if $\mathcal{C}$ is a triple complex or a complex of flower, $M \in \mathfrak{M}_1(G)$, and $M' \in \mathfrak{M}_2(G)$. In this case the statement we prove is clear. $\square$

Corollary 13. *Let complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ be such that $\mathcal{C}_2 = \{T\}$ and the cutset T splits $V(\mathcal{C}_1)$. Let the cutset T belong to a part $A \in \mathrm{Part}(\mathcal{C}_1)$, $R = \mathrm{Bound}(A)$ and $\overline{A}$ is a part of $\mathrm{Part}(R)$ not intersecting $\mathrm{Int}(A)$. Then $|\mathrm{Part}(R)| = |\mathrm{Part}(T)| = 2$. In addition, parts of $\mathrm{Part}(T)$ can be denoted by B and $\overline{B}$ such that the following statements hold.*

1) *$|\overline{A} \cap \overline{B}| = 2$.*

2) $\mathrm{Nb}(\overline{B}) = \overline{B} \subset \mathrm{Nb}(A)$.

3) *All cutsets of the complex $\mathcal{C}_1$ belong to the part $B \in \mathrm{Part}(\mathcal{C}_2)$.*

Proof. By lemma 29 the cutsets R and T are dependent, T separates a vertex $x \in R$ from other vertices of the set $V(\mathcal{C}_1)$ and both cutsets T and R can be

complemented by an edge xy (where $y \in T$). Moreover, $T \setminus A = \{y\}$. Since T does not belong to big complexes, $T \cap R = \varnothing$ and $|\mathrm{Part}(R)| = |\mathrm{Part}(T)| = 2$. Let $\mathrm{Part}(T) = \{B, \overline{B}\}$ where $x \in \overline{B}$. Let us check that all the statements hold.

1) Since $T \cap R = \varnothing$ we have $T \cap \overline{A} = \{y\}$ and $R \cap \overline{B} = \{x\}$. By corollary 2 that means $\overline{A} \cap \overline{B} = \{x, y\}$.

2) By lemma 26 all edges which complement T lie in the part $\overline{B}$, hence, $\mathrm{Nb}(\overline{B}) = \overline{B}$. In addition, $\overline{B} \setminus A = \overline{B} \cap \mathrm{Int}(\overline{A}) = \{y\} \subset \mathrm{Nb}(A)$, thus, $\overline{B} \subset \mathrm{Nb}(A)$.

3) Since $\mathrm{Nb}(\overline{B}) = \overline{B}$, then only cutsets contained in $\overline{B}$ can belong to the part $\overline{B}$. However, $V(\mathcal{C}_1) \cap \overline{B} \subset \{x, y\}$, hence, cutsets of the complex $\mathcal{C}_1$ cannot belong to the part $\overline{B}$. $\square$

Lemma 30. *For any maximal nontrivial cut M one of two following statements holds.*

1° *All* 3*-cutsets contained in M (i.e. inner sets and boundaries of M) belongs to some complex $\mathcal{C}$.*

2° *A vertex set of any complex $\mathcal{C}$ is contained in a neighborhood of some part of* $\mathrm{Part}(M)$*.*

Proof. By lemma 16 if a cutset $T \in \mathfrak{R}_3(G)$ is not contained in the neighborhood of any part of $\mathrm{Part}(M)$, then T with both boundaries of M generates a flower, which is contained in a maximal flower F. Then all inner sets and boundaries of the cut M belong to the complex of flower F, and statement 1° holds.

Let any cutset $T \in \mathfrak{R}_3(G)$ be contained in a neighborhood of some part of $\mathrm{Part}(M)$. We shall prove, that then statement 2° holds. In the case $|\mathcal{C}| = 1$ it is clear. Assume, that $|\mathcal{C}| > 1$ and consider several cases.

a. *Let $\mathcal{C}$ be a complex of big or small cut.* Note, that the vertex set of another cut M' is contained in the neighborhood of some part of $\mathrm{Part}(M)$. It is clear, since any two vertices of $V(M')$ are either adjacent, or belong to a 3-cutset contained in M' — in both cases they cannot lie in interiors of different parts of $\mathrm{Part}(M)$. Hence statement 2° for a complex of big or small cut immediately follows.

b. *Let $\mathcal{C}$ be a triple complex with line S.* Since the cut M is nontrivial, then S is independent with both boundaries of M. Let $S \subset G_1^M$. Then a vertex set of any cut with boundary S is contained in $\mathrm{Nb}(G_1^M)$, consequently, $V(\mathcal{C}) \subset \mathrm{Nb}(G_1^M)$.

c. *It remains to consider the case when $\mathcal{C}$ is the complex of a flower F.* In this case $V(F)$ is contained in the neighborhood of some part of $\mathrm{Part}(M)$ (let it be G_1^M), since any two vertices of the set $V(F)$ are either adjacent, or

belong to a 3-cutset. Hence, any set of the flower F is independent with T_2^M, consequently, G_2^M is contained in some part of $\mathrm{Part}(F)$ — let it be $G_{i,i+1}$. Thus, only $u_{i,i+1}$ can be a vertex of $V(\mathcal{C})$ not lying in $\mathrm{Nb}(G_1^M)$ (and, hence, contained in $\mathrm{Int}(G_2^M)$). It is possible when $u_{i,i+1} \neq p$. In this case $u_{i,i+1}$ is the only vertex of the set $G_{i,i+1}$ (and, consequently, of the set $G_2^M \subset G_{i,i+1}$) which is adjacent to p. Since $u_{i,i+1} \in \mathrm{Int}(G_2^M)$, we have, that $p \in T_2^M$. Thus p cannot be an end of edge $px \in M$: otherwise p is adjacent to $u_{i,i+1}$, x and only them, that is impossible. By lemma 5, hence, the cut M can be complemented by an edge $pu_{i,i+1}$. We obtain a contradiction with maximality of M. Consequently, $V(\mathcal{C}) \subset \mathrm{Nb}(G_1^M)$. □

Lemma 31. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two complexes. Then all cutsets of $\mathcal{C}_2 \setminus \mathcal{C}_1$ belong to one part $A \in \mathrm{Part}(\mathcal{C}_1)$. Moreover, $V(\mathcal{C}_2) \subset \mathrm{Nb}(A)$.*

Proof. In the case $|\mathcal{C}_2| = 1$ the statement of lemma is obvious. In the case $|\mathcal{C}_1| = 1$ this statement immediately follows from corollary 13. Thus it is enough to consider the case $|\mathcal{C}_1| > 1$ and $|\mathcal{C}_2| > 1$. Hence we obtain that the sets $V(\mathcal{C}_1)$ and $V(\mathcal{C}_2)$ does not contain each other. (The vertex set of a big complex by construction cannot be a subset of the vertex set of another complex. For a small complex it is possible only if it consists of one cutset, which is a boundary of big complex.) Consider a vertex $u \in V(\mathcal{C}_2) \setminus V(\mathcal{C}_1)$ and a part $A \in \mathrm{Part}(\mathcal{C}_1)$ containing u. Since $u \notin V(\mathcal{C}_1)$, we obtain, that $u \in \mathrm{Int}(A)$.

Note, that since $|\mathcal{C}_1| > 1$, the neighborhood of each part of $B \in \mathrm{Part}(\mathcal{C}_1)$ is contained in $V(\mathcal{C}_1) \cup B$. Hence no 3-cutset can intersect interiors of two parts of $\mathrm{Part}(\mathcal{C}_1)$, consequently, all cutsets of the complex $\mathcal{C}_2$ intersecting $\mathrm{Int}(A)$ belong to A.

Let us prove, that $V(\mathcal{C}_2) \subset \mathrm{Nb}(A)$. Consider several cases.

1. *Let $\mathcal{C}_2$ be a complex of small cut.* If both cutsets of this complex contain u, then these cutsets belong to A and, hence, are contained in $\mathrm{Nb}(A)$. If only one cutset $T \in \mathcal{C}_2$ contains u, then $V(\mathcal{C}_2)$ consists of vertices of the set T and a vertex u_1 adjacent to u. Since $u \in \mathrm{Int}(A)$ we obtain $u_1 \in A$.

2. *Let $\mathcal{C}_2$ be a big complex.*

2.1. *Let $\mathcal{C}_1$ be a complex of big or small cut.* Then the statement we prove follows from lemma 30.

2.2. *Let $\mathcal{C}_1$ be the triple complex with line S.* In this case the cutset S is independent with all cutsets of $\mathcal{C}_2$, hence, $V(\mathcal{C}_2)$ is contained in some part of $\mathrm{Part}(S)$, and every part of $\mathrm{Part}(S)$ is a neighborhood of correspondent part of $\mathrm{Part}(\mathcal{C}_1)$.

2.3. *Let $\mathcal{C}_1$ be the complex of a flower F.* Then $A = G'_{i,i+1}$ for some i. Consider the set $M'_{i,i+1}$. If $M'_{i,i+1}$ is a maximal cut, the statement we prove

immediately follows from lemma 30. This statement is also clear if $M'_{i,i+1}$ contains no edge (in this case it follows from lemma 18 and lemma 19, that $M'_{i,i+1}$ is a single cutset, which is the boundary of the part A). Hence it is enough to consider the case, when $M'_{i,i+1}$ contains at least one edge, but is not a maximal cut. It is clear from definition 20, that it is possible only when F is a nonsingular flower, exactly one of the parts $G_{i-1,i}$ and $G_{i+1,i+2}$ is empty (without loss of generality assume, that it is $G_{i+1,i+2}$) and the center p is adjacent to exactly one vertex of the part $G_{i,i+1}$ (denote this vertex by u'). Consider this case in details.

It follows from lemma 15, that the cut $M'_{i,i+1}$ can be complemented only by the edge pu'. Denote the resulting maximal cut by M. By lemma 30 the set $V(\mathcal{C}_2)$ is contained in the neighborhood of some part of $\mathrm{Part}(M)$. Note, that $G_{i,i+2} = \mathrm{Nb}(A)$ is a neighborhood of one part of $\mathrm{Part}(M)$. If $V(\mathcal{C}_2) \subset G_{i,i+2}$, then the statement we prove is fulfilled. The other part of $\mathrm{Part}(M)$ is $G_{i+2,i}$, and $\mathrm{Nb}(G_{i+2,i}) = G_{i+1,i} \cup \{u'\}$. Thus we can consider the case when $V(\mathcal{C}_2) \subset G_{i+1,i} \cup \{u'\}$ and $V(\mathcal{C}_2) \not\subset V(M)$. Note, that then $V(\mathcal{C}_2) \cap \mathrm{Int}(A) = \{u'\}$ and $u' = u$. Consider a cutset $T \in \mathcal{C}_2$ containing u. Since $\mathcal{C}_2$ is a big complex, there is a cutset $R \in \mathcal{C}_2$ dependent with T. Whence follows that $T \neq \{q_i, u, q_{i+1}\}$. But then T is dependent with $Q_{i,i+1}$, i.e. T splits $V(F)$. By lemmas 18 and 19 we have, that $T = \{q_i, u, q_{i+2}\}$, and, moreover, T is the only cutset of the complex $\mathcal{C}_2$ which contains u. Since R and T are dependent, then $q_{i+1} \in R$, i.e. $\{u, q_i, q_{i+1}, q_{i+2}\} \subset V(\mathcal{C}_2)$. In addition, it follows from lemma 11, that $q_{i+2} \notin R$.

Since the vertex u belongs to exactly one cutset of $\mathcal{C}_2$, this complex is not a complex of big cut. It remains to consider two cases: $\mathcal{C}_2$ is a complex of flower or a triple complex.

2.3.1. *Let $\mathcal{C}_2$ be the complex of a flower F'.* Clearly, in this case u is a petal of F' and the flower F' has four petals. In addition, $V(\mathcal{C}_2) = V(F')$, since otherwise $\mathcal{C}_2$ would be a complex of big cut. Note, that only q_i can be a center of F' (since another vertex cannot belong to both cutsets $T, R \in \mathcal{C}_2 = \mathfrak{R}(F')$). But then a cutset $R' = \{q_i, p, q_{i+1}\}$ contains the center of F' and is dependent with the cutset $T \in \mathfrak{R}(F)$. Hence by lemma 4 we have, that $R' \in \mathfrak{R}(F)$ and, since F' has only 4 petals, then $R' = R$ and $F' = (q_i; u, q_{i+1}, q_{i+2}, p)$. Thus $V(\mathcal{C}_2) \subset \mathrm{Nb}(A)$.

2.3.2. *Let $\mathcal{C}_2 = \mathcal{C}(N)$ be a triple complex, and a triple cut $N = M_1 \cup M_2 \cup M_3$ has line S.* The line of a triple complex splits the graph into three parts, hence by item 2 of lemma 21 it cannot split vertex set of the flower F. Then $S \neq T$. Moreover, S and T are independent, since otherwise three vertices of the set T would be in three different connected components of the graph $G - S$, i.e. the cutset S splits $\{q_i, q_{i+2}\} \subset V(F)$. Now without loss of generality we may assume, that the cutset T is contained in the

cut M_1'. Remind, that the vertex $u \in T$ do not belong to other cutsets of the complex $\mathcal{C}_2$. In particular, $u \notin S$. Thus there exists an edge $ux \in M_1'$, but in this case the vertex u belongs to at least two cutsets of the complex $\mathcal{C}_2$. We obtain a contradiction and show, that this case is impossible.

Thus we have proved, that in all cases $V(\mathcal{C}_2) \subset \mathrm{Nb}(A)$. Hence any cutset of the complex $\mathcal{C}_2$ either intersects $\mathrm{Int}(A)$ and, consequently, belongs to A, or is contained in $V(\mathcal{C}_1)$. Note, that a 3-cutset $R \subset V(\mathcal{C}_1)$ can either belong to the complex $\mathcal{C}_1$, or be a boundary of some part of $\mathrm{Part}(\mathcal{C}_1)$. Let a cutset $R \in \mathcal{C}_2$ be a boundary of a part $A_1 \in \mathrm{Part}(\mathcal{C}_1)$ different from A. Then R is independent with all sets of $\mathcal{C}_2$, that is possible only when $\mathcal{C}_2$ is a complex of small cut. In this case $V(\mathcal{C}_2) \subset \mathrm{Nb}(A_1)$, hence $V(\mathcal{C}_2) \cap \mathrm{Int}(A) = \varnothing$. We obtain a contradiction. $\square$

Remark 19. Note, that two complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ can have nonempty intersection. For example, complexes of two nondegenerate flowers can have common boundary cut.

Definition 32. Let $\mathfrak{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ be the set of all complexes of the graph G. Denote by $A_{i \supset j}$ the part of $\mathrm{Part}(\mathcal{C}_i)$ to which all cutset of the complex $\mathcal{C}_j$ belong. Let us say, that the complex $\mathcal{C}_j$ belongs to the part $A_{i \supset j}$.

For each complex $\mathcal{C}_i$ we denote by $\mathfrak{C}_i$ the decomposition of all other complexes into classes: two complexes $\mathcal{C}_j$ and $\mathcal{C}_\ell$ belongs to the same class of this decomposition if and only if $A_{i \supset j} = A_{i \supset \ell}$.

We say that a complex $\mathcal{C}_i$ *separates* $\mathcal{C}_j$ from $\mathcal{C}_\ell$, if they belong to different classes of the decomposition $\mathfrak{C}_i$. We call complexes $\mathcal{C}_i$ and $\mathcal{C}_j$ *neighboring*, if no other complex separates $\mathcal{C}_i$ from $\mathcal{C}_j$. Denote by $T(G)$ a hypergraph which vertices are complexes of the graph G, and hyperedges are all maximal with respect to inclusion sets of pairwise neighboring complexes. We call the hypergraph $T(G)$ a *hypergraph of decomposition* of G.

The construction of hypergraph of decomposition is described in details in [9, section 2]. In this work we will use the following theorem (see [9, theorem 3]).

Theorem 4. *Let for each element of the set V the decomposition of all other elements of V into classes correspond. Let for every $a, b, c \in V$ the following condition hold: if a separates b from c, then b does not separate a from c. Then the following statements hold.*

1) *The hypergraph of this decomposition $T(V)$ is a hypertree (i.e., any cycle of this hypergraph is a subset of some hyperedge).*

2) *Let for some vertex $a \in V$ the hypergraph $T(V) - a$ have connected components $W_1, \dots, W_\ell$. Then the element a decompose elements of $V \setminus \{a\}$ exactly into classes $W_1, \dots, W_\ell$.*

Lemma 32. *Let B be a nonempty part of* $\mathrm{Part}(\mathcal{C}_j)$*, different from* $A_{j\supset i}$*. Then* $\mathrm{Nb}(B) \subset \mathrm{Nb}(A_{i\supset j})$*. Moreover, if* $B \not\subset A_{i\supset j}$*, then* $|\mathcal{C}_j| = 1$ *and the only cutset of the complex* $\mathcal{C}_j$ *splits* $V(\mathcal{C}_i)$*.*

Proof. Let $R = \mathrm{Bound}(A_{i\supset j})$, $S = \mathrm{Bound}(A_{j\supset i})$, $T = \mathrm{Bound}(B)$.

Note, that if $B \subset A_{i\supset j}$, then $\mathrm{Nb}(B) \subset \mathrm{Nb}(A_{i\supset j})$. Indeed, if $y \in \mathrm{Nb}(B) \setminus B$, then the cutset T can be complemented by an edge xy (where $x \in T$). If in addition $y \notin A_{i\supset j}$, then $x \in R$, and it follows from lemma 5, that the cutset R can be also complemented by the edge xy. Hence $y \in \mathrm{Nb}(A_{i\supset j})$.

Thus, it is enough to prove, that $B \subset A_{i\supset j}$. We shall do it in all cases, except 1.3. Consider several cases.

1. *Let* $\mathcal{C}_j = \{T\}$. Clearly, then $S = T$. This case is divided into the following subcases.

1.1. $T = R$. Then T does not split $V(\mathcal{C}_i)$, hence, $V(\mathcal{C}_i) \subset A_{j\supset i}$. Moreover, in this case $\mathcal{C}_i$ is a big complex, consequently, $A_{i\supset j}$ is a union of all parts of $\mathrm{Part}(T)$ except $A_{j\supset i}$. Thus, $B \subset A_{i\supset j}$.

1.2. $T \neq R$ *and* T *does not split* $V(\mathcal{C}_i)$*.* Then $R \subset V(\mathcal{C}_i) \subset A_{j\supset i}$. Hence, $R \cap \mathrm{Int}(B) = \varnothing$, i.e. by lemma 2 we have, that R does not split B. Then B is contained in some part $H \in \mathrm{Part}(R)$. In addition $H \subset A_{i\supset j}$, since $T \subset A_{i\supset j}$ and $T \neq R$. Thus, $B \subset A_{i\supset j}$.

1.3. T *splits* $V(\mathcal{C}_i)$*.* In this case by corollary 13 we have, that $\mathrm{Part}(T) = \{A_{j\supset i}, B\}$ and $\mathrm{Nb}(B) = B \subset \mathrm{Nb}(A_{i\supset j})$. Note, that it is the only case when $B \not\subset A_{i\supset j}$.

2. *Let* $|\mathcal{C}_j| > 1$*.*

Then $V(\mathcal{C}_j) \not\subset V(\mathcal{C}_i)$, consequently, $V(\mathcal{C}_j) \cap \mathrm{Int}(A_{i\supset j}) \neq \varnothing$. Moreover, in this case $V(\mathcal{C}_i) \subset \mathrm{Nb}(A_{j\supset i}) \subset A_{j\supset i} \cup V(\mathcal{C}_j)$. Note, that the cutset T does not split $A_{j\supset i} \cup V(\mathcal{C}_j)$, consequently, $A_{j\supset i} \cup V(\mathcal{C}_j)$ is contained in some part $H \in \mathrm{Part}(T)$. In addition $H \neq B$, since $A_{j\supset i} \subset H$. Then $V(\mathcal{C}_i) \cap \mathrm{Int}(B) = \varnothing$ and B is contained in some part of $\mathrm{Part}(\mathcal{C}_i)$. We want to prove that $B \subset A_{i\supset j}$. For this purpose it is enough to check, that $T \cap \mathrm{Int}(A_{i\supset j}) \neq \varnothing$.

Suppose the converse. Then T either contains an inner vertex of another part of $\mathrm{Part}(\mathcal{C}_i)$, or T is a subset of $V(\mathcal{C}_i)$. Consider these two cases in details. Let $\overline{A_{j\supset i}}$ be the part of $\mathrm{Part}(S)$ which contains $V(\mathcal{C}_j)$.

2.1. *Let a part* $F \in \mathrm{Part}(\mathcal{C}_i)$ *such that* $F \neq A_{i\supset j}$ *and* $T \cap \mathrm{Int}(F) \neq \varnothing$ *exist.* It is possible only if $\mathcal{C}_i = \{R\}$ and R splits $V(\mathcal{C}_j)$. Then by corollary 13 we obtain, that $|F \cap \overline{A_{j\supset i}}| = 2$. On the other side, $T \subset V(\mathcal{C}_j) \subset \overline{A_{j\supset i}}$, hence, $T \not\subset F$. But then T and R are dependent, that is impossible, since $R \subset H \in \mathrm{Part}(T)$. Thus, this case is impossible.

2.2. *Let* $T \subset V(\mathcal{C}_i)$*.* Then $T \subset \mathrm{Nb}(A_{j\supset i})$. As we know, $S = \mathrm{Bound}(A_{j\supset i})$ and S is independent with T, consequently, there exists a cut M with boundaries S and T. In addition, $V(\mathcal{C}_j) \subset \overline{A_{j\supset i}} \cap H = V(M)$. Since $|\mathcal{C}_j| > 1$, hence

by the definition of complex it follows, that $V(\mathcal{C}_j) = V(M)$ and M is a maximal cut.

Since $V(\mathcal{C}_i) \subset H \in \mathrm{Part}(T)$, then the cutset T does not split $V(\mathcal{C}_i)$. Then it follows from $T \subset V(\mathcal{C}_i)$, that T is a boundary of the complex $\mathcal{C}_i$. Also note, that $\mathcal{C}_i \neq \{T\}$, since the cutset T belongs to different from $A_{j\subset i}$ part $B \in \mathrm{Part}(\mathcal{C}_j)$. Hence, $B \in \mathrm{Part}(\mathcal{C}_i)$ and $\mathrm{Nb}(B) = B \cup V(M)$. Consequently, $V(M) \subset V(\mathcal{C}_i)$ and we obtain a contradiction with $V(\mathcal{C}_j) \cap \mathrm{Int}(A_{i\supset j}) \neq \varnothing$. Thus, this case is also impossible. $\square$

Theorem 5. 1) *The hypergraph $T(G)$ is a hypertree (i.e., each cycle of $T(G)$ is a subset of some hyperedge).*

2) *Let $\mathcal{C}_i \in \mathfrak{C}$ and $H_1, \dots, H_\ell$ be connected components of the hypergraph $T(G) - \mathcal{C}_i$. Then $\mathfrak{C}_i = \{H_1, \dots, H_\ell\}$.*

Proof. Both statements of this theorem immediately follow from theorem 4, hence it is enough to verify the conditions of this theorem.

Suppose, that the complex $\mathcal{C}_i$ separates $\mathcal{C}_j$ from $\mathcal{C}_\ell$, i.e. $A_{i\supset j} \neq A_{i\supset \ell}$. We need to prove, that $\mathcal{C}_j$ does not separate $\mathcal{C}_i$ from $\mathcal{C}_\ell$, i.e. $A_{j\supset i} = A_{j\supset \ell}$. By lemma 32 we have $\mathrm{Nb}(A_{i\supset \ell}) \subset \mathrm{Nb}(A_{j\supset i})$, consequently, $V(\mathcal{C}_\ell) \subset \mathrm{Nb}(A_{j\supset i})$. On the other side, $V(\mathcal{C}_\ell) \subset \mathrm{Nb}(A_{j\supset \ell})$.

Let $A_{j\supset i} \neq A_{j\supset \ell}$. Then $V(\mathcal{C}_\ell) \subset \mathrm{Nb}(A_{j\supset i}) \cap \mathrm{Nb}(A_{j\supset \ell})$. In addition, $\mathrm{Nb}(A_{j\supset i}) \neq A_{j\supset i}$, since otherwise $V(\mathcal{C}_\ell) \subset A_{j\supset i}$ and, consequently, the complex $\mathcal{C}_\ell$ belongs to the part $A_{j\supset i}$. Further we consider the following two cases.

1. *Let* $|\mathcal{C}_j| > 1$. Then $V(\mathcal{C}_\ell) \subset \mathrm{Nb}(A_{j\supset i}) \cap \mathrm{Nb}(A_{j\supset \ell}) \subset V(\mathcal{C}_j)$. Hence, $\mathcal{C}_j$ is a big complex and $\mathcal{C}_\ell = \{T\}$ where $T = \mathrm{Bound}(A_{j\supset \ell})$. That is, $V(\mathcal{C}_\ell) \subset A_{j\supset \ell}$. Moreover, by lemma 32 in this case $A_{j\supset \ell} \subset A_{i\supset j}$, hence, $V(\mathcal{C}_\ell) \subset A_{i\supset j}$. But then the complex $\mathcal{C}_\ell$ belongs to the part $A_{i\supset j}$, i.e. $A_{i\supset j} = A_{i\supset \ell}$. We obtain a contradiction.

2. *Let* $\mathcal{C}_j = \{R\}$. Since $\mathrm{Nb}(A_{j\supset i}) \neq A_{j\supset i}$, then the cutset R is nonsingle. Hence, $\mathrm{Part}(\mathcal{C}_j) = \{A_{j\supset i}, A_{j\supset \ell}\}$ and by lemma 26 we obtain, that all edges which complement the cutset R lie in the part $A_{j\supset \ell}$. Thus $\mathrm{Nb}(A_{j\supset \ell}) = A_{j\supset \ell}$, i.e. $V(\mathcal{C}_\ell) \subset A_{j\supset \ell}$. Hence R splits $V(\mathcal{C}_i)$, since otherwise by lemma 32 we have $A_{j\supset \ell} \subset A_{i\supset j}$ and, consequently, the complex $\mathcal{C}_\ell$ belongs to the part $A_{i\supset j}$, that contradicts the assumption. Let $S = \mathrm{Bound}(A_{i\supset j})$. Since R splits $V(\mathcal{C}_i)$, then by corollary 13 we have $\mathrm{Part}(S) = \{A_{i\supset j}, \overline{A_{i\supset j}}\}$ and $|A_{j\supset \ell} \cap \overline{A_{i\supset j}}| = 2$. However, $|V(\mathcal{C}_\ell)| \geq 3$, consequently, $V(\mathcal{C}_\ell) \cap \mathrm{Int}(A_{i\supset j}) \neq \varnothing$ and the complex $\mathcal{C}_\ell$ belongs to the part $A_{i\supset j}$. We have a contradiction. $\square$

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